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# THE JET-CHAIN- AND THE JET-WAVE-VIBRATOR

BY

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## PREFACE

In previous papers and treatises the properties of the socalled jet-wave have been discussed and a series of its applications have been described[1]. In all the cases referred to the jet-wave was produced from an electrically conductive liquid jet, preferably a mercury jet, and the conductivity was not merely an essential condition for the production of the wave but also for the various applications. (The jet-wave interruptor, the jet-wave commutator, the jet-wave oscillograph). At a certain stage it occurred to the author that a new application of possibly far-reaching consequences might be made of a periodic jet-wave by using it for the production of a vibratory motion synchronous with the wave. And so the investigation dealt with in the present paper was taken up after some preliminary observations on the

[1] 1. Nye Ensrettere og periodiske Afbrydere. København 1918.

2. Development of the Jet-Wave Rectifier, "Engineering". September 9. and 16. 1927.

3. Den konstruktive Udvikling af Straalebølgeensretteren. Elektroteknikeren Nr. 23. 1927.

4. Güntherschulze. Die konstruktive Durchbildung des Quecksilber-Wellenstrahl-Gleichrichters. Elektrotechnische Zeitschrift, 1928 pg. 1224.

5. The Jet-Wave and its Applications, "Engineering" Sept. 14. 1928.

6. Theory of the Jet-Wave. Vidensk. Selsk. math.-fys. Medd. IX, 2. 1919.

said motion had been made. It proved impossible — as usual — to solve the differential-equations involved in even the simplest case of motion (the see-saw motion) in terms of available functions. With a view to orientation the wave was therefore provisionally replaced by a simpler but similar system, i. e. by that of the twin jet-chain. The latter is by no means a purely abstract conception, on the contrary it may easily be produced and seems per se adaptable to a good many practical applications. The investigations on the motion produced by jet-chains are found in the first chapter of the present paper. — It was, however, found that a very characteristic observation pertaining to the original system with an ordinary jet-wave did not find its explanation by replacing the wave by the twin jet-chain. The observation referred to consisted in the fact of a simple see-saw, without external controlling or directive moment, exhibiting a fictive directive moment keeping the see-saw vibrating, under the influence of the wave, about a position perpendicular to the axis of the latter. This observation was for some time found very puzzling. It could be shown that with a regular periodic wave of constant amplitude no such fictive directive moment would occur. Eventually it turned out that the observed quality of the see-saw motion could be carried back to that property of the ordinary jet-wave consisting in its amplitude increasing steadily with the distance from the starting-point of the wave. Especially could it be shown that the fictive directive moment was particularly pronounced with the electromagnetically produced wave. The second chapter of the present paper deals with the relations here referred to. It is believed that the contents of the two chapters will prove a suitable means for the discussion of various practical applications of the motion considered.

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# I.

# The Jet-Chain-Vibrator.

## 1. The damping Effect of a Liquid-Jet.

In fig. 1a $D$ designates a disk which is hit by a liquid jet $J$ with a mass per cm $m$ and a velocity $v$. The liquid of the jet will be reflected in the shape of a nearly circular plane film fig. 2. Thus the particles of the jet will, during the collision with $D$, lose their total forward velocity $v$ and consequently they will act on $D$ with a force

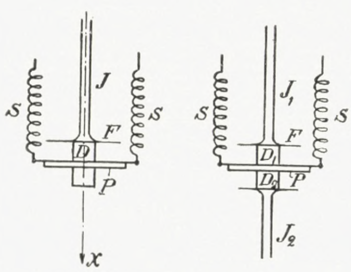


Fig. 1 a—b. Oscillatory Systems hit by Jets.

$$(1) \qquad F_0 = m v^2,$$

that is to say, if $D$ is at rest relatively to the nozzle of $J$. If $D$ has itself a volocity $\frac{dx}{dt}$ in the direction of $J$, the force will be

$$(2) \qquad F = m\left(v - \frac{dx}{dt}\right)^2$$

thus smaller or greater than $F_0$ according as $\frac{dx}{dt}$ is positive or negative i. e. going in the direction of or against the motion of $J$.

If $\frac{dx}{dt}$ is small compared to $v$, we may transform (2) into

$$F = m v^2 - 2 mv \cdot \frac{dx}{dt} \tag{3}$$

from which it is seen that the influence of $J$ on the motion of $D$ is that of a driving force $mv^2$ combined with a

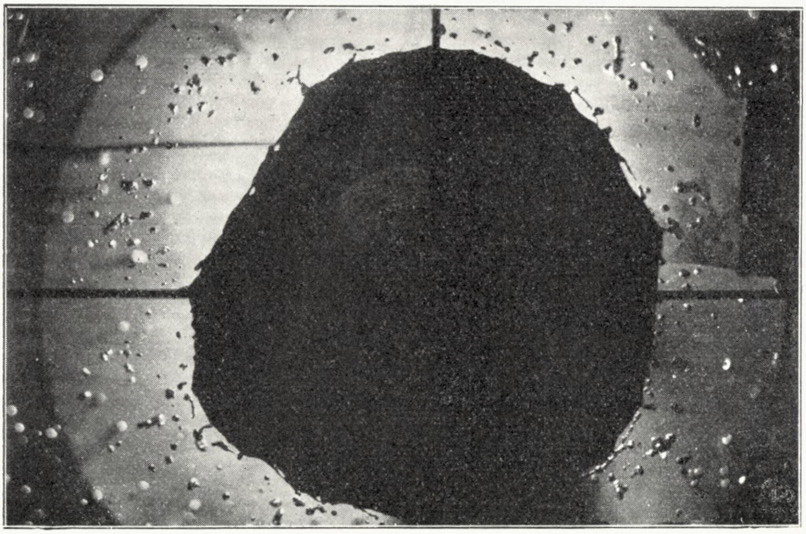

Fig. 2. Mercury Jet-Film.

damping force $2 mv \frac{dx}{dt}$. If two jets $J_1 J_2$ fig. 1b of the same velocity and mass per cm hit the system $D_1 D_2$ from opposite sides the two driving forces compensate each other, while the damping forces are added. This is not only true in case of $\frac{dx}{dt}$ being small compared to $v$, but it holds good in any case. For the resultant force originating from the two jets is obviously

$$F = m\left(v - \frac{dx}{dt}\right)^2 - m\left(v + \frac{dx}{dt}\right)^2 = -4 mv \frac{dx}{dt}. \tag{4}$$

The twin-jet system in fig. 1b thus constitutes a means for the introduction of a damping of a definite and easily calculable size. If especially the twin-jet damper is applied to an oscillatory system like that in fig. 1b the motion of this latter system will be determined by

$$(5) \qquad m_0 \frac{d^2 x}{dt^2} + (p + 4\, mv) \frac{dx}{dt} + kx = X,$$

$m_0$ being the mass, $p$ the damping factor and $k$ the directive force of the system, while $X$ stands for the driving force. Provided

$$(6) \qquad (p + 4\, mv)^2 < 4\, km_0$$

the motion will be that of damped vibrations with a period approximately equal to

$$(7) \qquad T = 2\pi \sqrt{\frac{m_0}{k}}$$

and with an amplitude

$$(8) \qquad A = A_0 \cdot e^{-\frac{p + 4\, mv}{2\, m_0} t} = A_0\, e^{-\alpha t}.$$

In order to convey an idea of the effectiveness of the damping device we may consider a system of which $m_0 = 1000\, g$, $k = 10^6$. The period will be $\frac{2\pi}{10\sqrt{10}}$ or ab. $\frac{2}{10}$ sec. We will assume the system to be hit by two mercury-jets with a velocity 700 cm/sec. and a diameter 0.5 cm. The mass $m$ per cm will then be $\frac{\pi}{4} \cdot \frac{1}{4} \cdot 13.6 = 2.67$ g/cm and $mv = 1.87 \cdot 10^3$ g/sec. If $p$ is negligible the condition (6) is fulfilled $(4\, mv)^2$ being $56 \cdot 10^6$ while $4\, km_0 = 4 \cdot 10^9$. Furthermore $\alpha = \frac{2\, mv}{m_0} = 3.74$. Thus during one period the amplitude is reduced to $e^{-3.74 \cdot 0.2} = 0.47$ and in 6 periods to $e^{-4.45} = 0.01$ of its original value.

## 2. The Nature of the Jet-Damper.

An investigation carried out with mercury-jets hitting a disk with a diameter somewhat greater than that of the jet showed that the radial velocity of the reflected mercury particles was very nearly equal to the velocity of the jet if the disk was at rest relatively to the nozzle of the jet. From this we conclude that if the disk has a velocity $\frac{dx}{dt}$ in the same direction as the jet, the radial velocity of the reflected particles will be equal to the relative velocity $v - \frac{dx}{dt}$. This conclusion is supported through the following consideration.

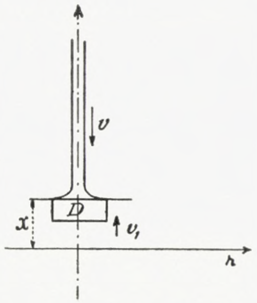

Fig. 3. Oscillatory System hit by a Jet.

If the disk $D$, fig. 3, is moving up against the jet with the velocity $v_1$ it will be acted on by a force $m(v+v_1)^2$. During one sec. $D$ will meet a quantity of liquid equal to $m(v+v_1)$ and it will supply a work

$$W = m(v+v_1)^2 \cdot v_1, \tag{1}$$

to the said mass of liquid. This work will have its equivalent in the excess of kinetic energy with which the liquid leaves $D$. The said excess is obviously

$$E = \frac{1}{2} m(v+v_1) \cdot (u^2 + v_1^2) - \frac{1}{2} m(v+v_1) v^2 \tag{2}$$

$u$ indicating the radial velocity of the liquid leaving the disk. To understand the second term $v_1^2$ in the brackets it must be noted that the reflected liquid receives the velocity of $D$ during the collision. Equalizing $E$ and $m(v+v_1)^2 \cdot v_1$ we get

$$u = v + v_1. \tag{3}$$

In just the same way it is found that $u = v - v_1$ if $D$ is moving in the same direction as the jet with the velocity $v_1$. Only in this case it is the jet which supplies work to the disk, consequently losing kinetic energy.

Having thus derived the relation (3) we may examine the shape and motion of the film in the case of the disk performing a simple harmonic motion

$$x = x_0 \sin \omega t. \tag{4}$$

We consider the film as consisting of particles which move on independently of each other. The motion of such a particle will, when it has left $D$, be determined by the two sets of equations

$$\frac{d^2x}{dt^2} = 0, \quad \frac{dx}{dt} = c_1, \quad x = c_1 t + c_2. \tag{5}$$

$$\frac{d^2r}{dt^2} = 0, \quad \frac{dr}{dt} = b_1, \quad r = b_1 t + b_2. \tag{6}$$

If the particle in question leaves $D$ at the moment $t_0$ we have

$$\begin{aligned} c_1 &= x_0 \omega \cos \omega t_0, \qquad b_1 = v + x_0 \omega \cos \omega t_0, \\ c_2 &= x_0 \sin \omega t_0 - x_0 \omega \cos \omega t_0 \cdot t_0, \\ b_2 &= -(v + x_0 \omega \cos \omega t_0) \cdot t_0. \end{aligned}$$

Thus

$$x = x_0 \sin \omega t_0 + x_0 \omega \cos \omega t_0 \cdot (t - t_0). \tag{7}$$

$$r = (v + x_0 \omega \cos \omega t_0) \cdot (t - t_0). \tag{8}$$

The equation of the curve of intersection between the film and a plane through the axis of the jet — and of $D$ — at the moment $t$ would be obtained if we could eliminate $t_0$ between (7) and (8). The practical way to find the curve of profile is to fix a value for $t$, say $2T\left(\omega = \frac{2\pi}{T}\right)$, and then to calculate $x$ and $r$ corresponding to a series of values of $t_0$. The result of such a determination has been reproduced in fig. 4. The abscissae are here the distances from the axis of the re-

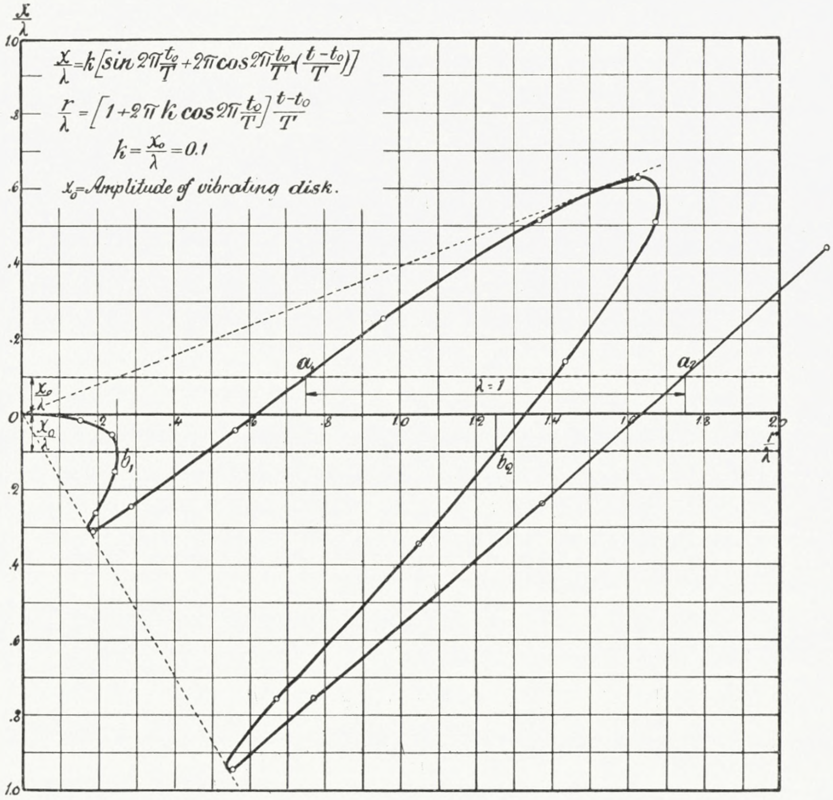


Fig. 4. Profile Curve of Film.

flecting disk measured in terms of the wave-length $\lambda = vT$, while the ordinates are the „deflection" of the film also measured in terms of $\lambda$. The curve may thus be taken to represent the system of equations

(7a) $$\frac{x}{\lambda} = k\left[\sin 2\pi\frac{t_0}{T} + 2\pi \cos 2\pi\frac{t_0}{T}\cdot\frac{t - t_0}{T}\right]$$

(8a) $$\frac{r}{\lambda} = \left[1 + 2\pi k \cos 2\pi\frac{t_0}{T}\right]\cdot\frac{t - t_0}{T}$$

$k$ designating $\frac{x_0}{\lambda}$ and being assumed in the case considered to have the value 0.1. In the figure the two lines $\frac{x_0}{\lambda}$ mark the extreme positions of the disk. In these positions the disk will reflect the liquid along lines perpendicular to the axis of the disk and with a velocity equal to $v$. We shall therefore expect to find particles of the film in the said lines with regular intervals of $\lambda$. In fig. 4 $a_1 a_2 \ldots b_1 b_2 \ldots$ represent such particles.

The character of the jet-damper is now fairly clear. The damping originates from a radiation of kinetic energy and the carrier of this energy is the reflected jet. It is easy to derive an expression for the energy radiated per period or per second. The kinetic energy supplied to the reflected liquid in the time $dt$ was

$$(9) \qquad dE = \frac{1}{2} m \left(v + \frac{dx}{dt}\right) \left[\left(v + \frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 - v^2\right] dt$$

$$= m\left(v + \frac{dx}{dt}\right)^2 \cdot \frac{dx}{dt} \cdot dt.$$

Remembering that $x = x_0 \sin \omega t$ and $\frac{dx}{dt} = x_0 \omega \cos \omega t = u_0 \cos \omega t$ and integrating over a period $T$ we find

$$(10) \qquad E_T = m v x_0^2 \omega^2 T = m v u_0^2 T$$

or the energy radiated per sec.

$$(11) \qquad E = m v u_0^2.$$

The same expression is obtained by considering the work supplied by the oscillatory system. In the time $dt$ this work is just represented by the last equation (9), $m\left(v + \frac{dx}{dt}\right)^2$ being the force with which the mercury jet is acted on and $\frac{dx}{dt} \cdot dt$ the way through which the force is acting. Finally we may derive (10) or (11) by remembering that the jet gives rise to a damping force $mv \cdot \frac{dx}{dt}$. The work done against this force in the time $dt$ is obviously

$$(12) \qquad dE = mv \cdot \frac{dx}{dt} \cdot \frac{dx}{dt} \cdot dt = mv\left(\frac{dx}{dt}\right)^2 \cdot dt .$$

Introducing $\frac{dx}{dt} = u_0 \cos \omega t$ and integrating over a period we get the expression (10).

### 3. Motion of a Body hit by the Jet.

We may now consider certain simple types of motion produced by the jet. We shall first think of a circular disk hit by a jet passing along the axis of the disk, fig. 3. The motion of the disk is determined by

$$(1) \qquad m_0 \frac{d^2x}{dt^2} = m\left(v - \frac{dx}{dt}\right)^2$$

if we assume that no frictional forces are acting. We solve this equation by putting $\frac{dx}{dt} = z$, thus getting

$$(2) \qquad m_0 \frac{dz}{dt} = m\,(v - z)^2 .$$

The solution is

$$(3) \qquad \frac{m_0}{v - z} = mt + c_1$$

from which

$$(4) \qquad \frac{dx}{dt} = v - \frac{m_0}{mt + c_1}$$

from which again by integrating we get

$$(5) \qquad x = vt - \frac{m_0}{m} \log \text{nat}\,(mt + c_1) + c_2 .$$

If $x = x_0$ and $\frac{dx}{dt} = u_0$ at the moment $t = 0$ we find $c_1 = \frac{m_0}{v - u_0}$ and $c_2 = x_0 + \frac{m_0}{m} \log \text{nat} \frac{m_0}{v - u_0}$, thus

$$(6) \qquad \frac{dx}{dt} = v - \frac{m_0}{mt + \frac{m_0}{v - u_0}}$$

and

$$(7) \qquad x = x_0 + vt - \frac{m_0}{m} \log \text{nat} \left(1 + \frac{m}{m_0}(v - u_0)\, t\right).$$

As an example we shall consider a projectile of mass $m_0 = 20$ g flying with a velocity $u_0 = 100000$ cm/sec against a mercury jet with a velocity 700 cm/sec and a diameter 0.5 cm. The mass per cm of this jet was found to be 2.67 g/cm thus $mv = 1.87 \cdot 10^3$. From (6) we find the time it takes to stop the projectile. Putting $\frac{dx}{dt} = 0$ we get

$$(8) \qquad t = \frac{m_0}{mv} \cdot \frac{u_0}{u_0 - v}$$

and in the case considered $t = 10.8 \cdot 10^{-3}$ sec or ab. $\frac{1}{100}$ sec. Introducing (8) in (7) we furthermore get the distance through which the projectile will fly before losing its velocity:

$$(9) \qquad x = \frac{m_0}{m} \cdot \frac{u_0}{u_0 - v} - \frac{m_0}{m} \log \text{nat} \left(1 - \frac{u_0}{v}\right)$$

which in our case gives $x = 7.5\,(1.007 - 4.969) = -29.8$ cm. The projectile will thus be stopped within 30 cm and in about $\frac{1}{100}$ sec.

It is still possible to solve the problem if the motion of the body hit by the jet is subject to a frictional force proportional to its velocity. In this case the differential equation is

$$(10) \qquad m_0 \frac{d^2x}{dt^2} + p \cdot \frac{dx}{dt} = m \left(v - \frac{dx}{dt}\right)^2.$$

Again we put $\frac{dx}{dt} = z$ thus reducing (10) to

$$m_0 \frac{dz}{dt} + pz = m\,(v - z)^2 \tag{11}$$

which may be written.

$$\frac{dz}{z^2 - 2\left(v + \frac{p}{2m}\right) z + v^2} = \frac{m}{m_0} \cdot dt \tag{12}$$

the integral of which is

$$\frac{1}{2\sqrt{\left(v + \frac{p}{2m}\right)^2 - v^2}} \log \text{nat} \frac{\sqrt{\left(v + \frac{p}{2m}\right)^2 - v^2} + \left(v + \frac{p}{2m}\right) - z}{\sqrt{\left(v + \frac{p}{2m}\right)^2 - v^2} - \left(v + \frac{p}{2m}\right) + z} = \frac{m}{m_0} t + c_1. \tag{13}$$

We may introduce abbreviations and write (13) thus

$$\frac{1}{A} \log \text{nat} \frac{B - z}{D + z} = Et + c_1 \tag{14}$$

from which we get

$$z = B - (D + B) \frac{e^{AEt + Ac_1}}{1 + e^{AEt + Ac_1}} = \frac{dx}{dt} \tag{15}$$

the solution of which is

$$x = Bt - \frac{D + B}{AE} \log \text{nat} \left(1 + e^{AEt + Ac_1}\right) + c_2. \tag{16}$$

If $x = 0$ and $\frac{dx}{dt} = 0$ at the moment $t = 0$ we find

$$c_1 = \frac{1}{A} \log \text{nat} \frac{B}{D} \text{ and } c_2 = \frac{D + B}{AE} \log \text{nat} \left(1 + \frac{B}{D}\right)$$

and

$$x = Bt - \frac{D + B}{AE} \log \text{nat} \frac{1 + \frac{B}{D} e^{AEt}}{1 + \frac{B}{D}}. \tag{17}$$

If $p$ is small compared to $2\,mv$ we get from (17)

$$x = vt - \frac{m_0}{m} \log \text{nat} \frac{1 - \left(1 + 2\sqrt{\frac{p}{mv}}\right) \cdot e^{2\sqrt{\frac{pv}{m}} \cdot \frac{m}{m_0} t}}{-2\sqrt{\frac{p}{mv}}}. \tag{18}$$

For $p=0$ the last factor above assumes the shape log nat $\frac{0}{0}$. Its actual value is found to be $1+\frac{mv}{m_0}t$, (18) being thus reduced to

$$x = vt - \frac{m_0}{m}\log\text{nat}\left(1+\frac{mv}{m_0}t\right) \tag{19}$$

in agreement with (7) for $u_0=0$ and $x_0=0$.

Finally we may think of the body as forming part of a complete oscillatory system with a directive force $kx$. The equation of the motion then becomes

$$m_0\frac{d^2x}{dt^2}+p\frac{dx}{dt}+kx = m\left(v-\frac{dx}{dt}\right)^2. \tag{20}$$

This equation will probably prove rather intricate unless it is assumed that $\frac{dx}{dt}$ is small compared to $v$. If this is the case, we may write $m\left(v-\frac{dx}{dt}\right)^2 = mv^2-2mv\frac{dx}{dt}$ and reduce (20) to

$$m_0\frac{d^2x}{dt^2}+(p+2mv)\frac{dx}{dt}+kx-mv^2=0 \tag{21}$$

thus to the well-known linear differential equation with constant coefficients. The problem of the motion is then easily solved and it may be solved not only in the case of no external forces acting on the oscillatory system but also for a good many cases of such forces.

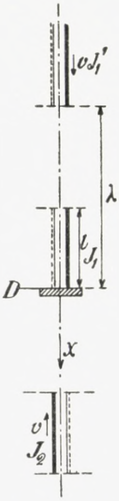


Fig. 5. The Translatory Jet-Chain-Vibrator.

### 4. The Jet-Chain-Vibrator, translatory Type.

We may now imagine a disk $D$, fig. 5, hit centrally and perpendicularly by two series of jet-pieces $J_1$ and $J_2$. We may term each of the series a jet-chain. The one of them $J_1 J_1'$ ... is moving downwards with the velocity $v$, the other $J_2 J_2'$ upwards with the same velocity. In order to avoid confusion the two series have in the figure been drawn sideways to the axis of $D$. Actually we shall think of them as travelling in the said axis.

Each of the jet-pieces will, in colliding with $D$, give rise to an impulse. Every second impulse is directed upwards, every second downwards. Thus $D$ will assume a vibratory motion. If the displacement $x$, the velocity $\frac{dx}{dt}$ and the forces are considered positive in the downward direction the motion due to a jet-piece such as $J_1$ is determined by

$$(1) \qquad m_0 \frac{d^2x}{dt^2} + p\frac{dx}{dt} + kx = m\left(v - \frac{dx}{dt}\right)^2 = mv^2\left(1 - \frac{2}{v}\cdot\frac{dx}{dt}\right)$$

and that due to a piece such as $J_2$ by

$$(2) \qquad m_0 \frac{d^2x}{dt^2} + p\frac{dx}{dt} + kx = -m\left(v + \frac{dx}{dt}\right)^2 = -mv^2\left(1 + \frac{2}{v}\cdot\frac{dx}{dt}\right),$$

$\frac{dx}{dt}$ being considered small compared to $v$.

Assuming $k = 0$ and furthermore assuming that $x = x_0$, $\frac{dx}{dt} = u_0$ at the moment of collision $t = 0$ we find for the motion due to a $J_1$-piece

$$(3) \qquad \frac{x - x_0}{\lambda/2} = \frac{1}{\alpha+2}\cdot\left(\frac{t}{T/2}\right) - \frac{\gamma}{\alpha+2}\left(U_0 - \frac{1}{\alpha+2}\right)\left(e^{-\frac{\alpha+2}{\gamma}\frac{t}{T/2}} - 1\right).$$

$$(4) \qquad \frac{dx}{dt} = v\left[\frac{1}{\alpha+2} + \left(U_0 - \frac{1}{\alpha+2}\right)e^{-\frac{\alpha+2}{\gamma}\cdot\frac{t}{T/2}}\right].$$

and for a $J_2$ piece

$$(5) \qquad \frac{x - x_0}{\lambda/2} = -\frac{1}{\lambda+2}\left(\frac{t}{T/2}\right) - \frac{\gamma}{\alpha+2}\left(U_0 + \frac{1}{\alpha+2}\right)\left(e^{-\frac{\alpha+2}{\gamma}\frac{t}{T/2}} - 1\right).$$

$$(6) \qquad \frac{dx}{dt} = v\left[-\frac{1}{\alpha+2} + \left(U_0 + \frac{1}{\alpha+2}\right)e^{-\frac{\alpha+2}{\gamma}\cdot\frac{t}{T/2}}\right].$$

Here $U_0 = \frac{u_0}{v}$ and $\alpha$, $\gamma$ and $\lambda$ are defined by $p = \alpha \cdot mv$, $m_0 = \gamma \cdot m\frac{\lambda}{2}$, while $\lambda$ is the chain-length, compare fig. 5, and $T = \frac{\lambda}{v}$. By means of (3)—(6) we may calculate the motion of $D$. In so doing we must consider separately each of the parts or phases of which the motion is built up and we must carefully account for the position and velocity with which each phase is concluded. In fig, 6a—d a review of the first four phases has been given. In fig. 6a the start-moment is presented. $D$ is in the zero-position

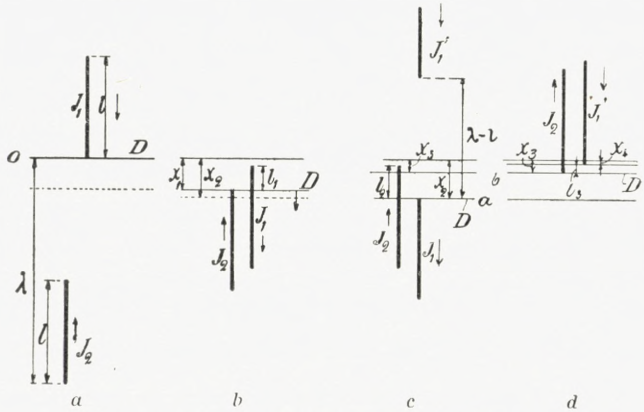


Fig. 6. Review of Phases of the Motion.

and has just been reached by $J_1$. It moves downwards and after the lapse of the time $t_1$ collides with $J_2$ which, coming from below, meets $D$. The latter is at that moment at a distance $x_1$ away from the starting-position. The relation between $x_1$ and $t_1$ is given by the formula.

(7) $$\lambda - l - x_1 = vt_1.$$

Fig. 6b gives the condition at the moment $t_1$. $J_1$ has not completely passed $D$ or rather been reflected from it. Thus during the phase beginning at $t_1$, $D$ is acted on both by a downward and upward force. Its motion is determined by

(8) $$m_0 \frac{d^2x}{dt^2} + p \frac{dx}{dt} = m\left(v - \frac{dx}{dt}\right)^2 - m\left(v + \frac{dx}{dt}\right)^2 = -4\,mv\frac{dx}{dt}$$

the solution of which is

$$(9)\qquad \frac{x-x_0}{\lambda/2} = -U_0 \cdot \frac{\gamma}{\alpha+4}\left(e^{-\frac{\alpha+4}{\gamma}\cdot\frac{t}{T/2}}-1\right)$$

$$(10)\qquad \frac{dx}{dt} = U_0 \cdot e^{-\frac{\alpha+4}{\gamma}\cdot\frac{t}{T/2}},$$

if $x = x_0$, $\frac{dx}{dt} = u_0$ at the beginning $t = 0$ of the phase.

The second phase ends at the moment $t_2$ and the position $x_2$. The relation between $t_2$ and $x_2$ is given by the formulae

$$(11)\qquad l_1 = 2l + 2x_1 - \lambda$$

$$(12)\qquad vt_2 = l_1 + (x_2 - x_1) = 2l + x_1 + x_2 - \lambda$$

both referring to fig. 6b. The situation after this is that indicated in fig. 6c. $J_1$ has completely passed $D$ which now in the following third phase is exclusively acted on by $J_2$. The motion is thus determined by the equations (5) and (6). It lasts till the moment $t_3$ corresponding to the position $x_3$ indicated in fig. 6d. The relation between $t_3$ and $x_3$ is given by

$$(13)\qquad l_2 = vt_0 + (x_2 - x_1) = 2l + 2x_2 - \lambda,$$

$$(14)\qquad vt_3 = l - l_2 + x_2 - x_3 = \lambda - l - x_2 - x_3$$

referring to fig. 6c. In fig. 6d it is assumed that $D$ has not yet reached $J_1'$ at the moment $t_3$ at which $J_2$ has passed $D$. If this assumption proves to hold good, a phase sets in, the fourth phase, during which $D$ is not acted on by any piece of jet. Its motion will therefore be given by

$$(15)\qquad m_0\frac{d^2x}{dt^2} + p\frac{dx}{dt} = 0$$

the solution of which may be written

$$(16)\qquad \frac{x-x_0}{\lambda/2} = -U_0\frac{\gamma}{\alpha}\left(e^{-\frac{\alpha}{\gamma}\cdot\frac{t}{T/2}}-1\right)$$

provided $x = x_0$ and $\dfrac{dx}{dt} = u_0$ at the beginning of the phase $t = 0$. In the way indicated the motion may be built up from phase to phase. By way of illustration the

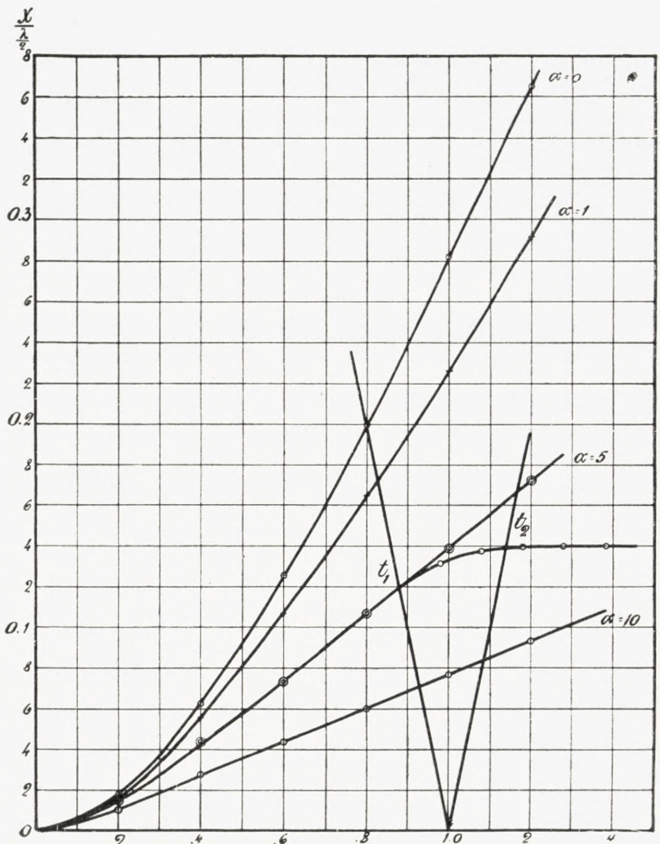


Fig. 7. Motion of the Translatory Vibrator.

first phase has been reproduced in fig. 7 corresponding to a series of values of $\alpha$. It has been assumed that the length of the jet-pieces is $\dfrac{\lambda}{2}$ and that $\gamma = 1$. The straight line which determines the end of the first phase is, according to fig. 6a, given by

$$(17) \qquad 1 - \frac{x_1}{\lambda/2} = \frac{t_1}{T/2}.$$

Furthermore the second phase corresponding to $\alpha = 5$ is indicated. It is seen that the velocity obtained by $D$ during the first phase is lost very soon due to the comparatively heavy damping. The velocity is however not completely reduced to naught at the moment $t_2$ which is determined by the point of intersection between the curve and a straight line

$$\frac{x_2}{\lambda/2} = \frac{t_2}{T/2} - 0.120. \tag{18}$$

After this the action of $J_2$ will turn the velocity and nearly, but only nearly, carry $D$ back to the starting-position. From what has been set forth it may already be concluded that the motion will consist of a series of vibrations displaced laterally with regard to the starting-position. The lateral displacement is of course due to the first impulse giving $D$ a downward deviation which cannot be compensated by the following impulses.

### 5. The Jet-Chain See-Saw.

We shall now consider a jet hitting one end of a balance or see-saw with the moment of inertia $J$, fig. 8. The jet has the velocity $v$ and the mass $m$ per cm. It meets the see-saw at a moment at which the latter has an angular velocity $\frac{d\theta}{dt}$ and is deflected $\theta$ from the normal position perpendicular to the jet. The velocity of the hitting point is consequently $v' = \frac{a}{\cos\theta} \cdot \frac{d\theta}{dt}$. During the time-interval $dt$ the bar is hit by the mass $mdt\,(v - v'\cos\theta)$ and the relative velocity perpendicular to the bar,

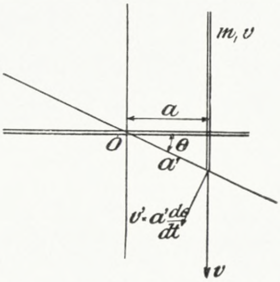


Fig. 8. Jet hitting on End of a See-Saw.

thus $(v \cos \theta - v')$, is destroyed. This means that the bar is acted on by a perpendicular force

$$(1) \qquad F = m\,(v - v' \cos \theta)\;(v \cos \theta - v')$$

and by a turning moment

$$(2) \qquad M = F\frac{a}{\cos\theta} = \frac{ma}{\cos^2\theta}\left(v - a\frac{d\theta}{dt}\right)\left(v\cos^2\theta - a\frac{d\theta}{dt}\right).$$

We shall assume $\theta$ always to be small enough to justify us in putting $\cos^2\theta = 1$. We then simply have

$$(3) \qquad M = ma\left(v - a\frac{d\theta}{dt}\right)^2$$

or if $a\dfrac{d\theta}{dt}$ is small compared to $v$

$$(4) \qquad M = mav^2\left(1 - 2\frac{a}{v}\cdot\frac{d\theta}{dt}\right).$$

Obviously the problem of the motion of the bar is now just the same as that of the motion of the disk above. Thus if the see-saw is acted on by a damping moment $-p\dfrac{d\theta}{dt}$ and by a directive moment $-h\theta$, the differential equation of its motion is

$$(5) \qquad I\frac{d^2\theta}{dt^2} + (p + 2\,ma^2v)\cdot\frac{d\theta}{dt} + (h\theta - mav^2) = 0.$$

Replacing here $I$ by $m_0$, $\theta$ by $x$, $a$ by 1 and $k$ by $h$ we have come back to equation (21) on pg. 16.

We may now study the motion of the see-saw when the latter is alternately hit by the jet-pieces $J_1\,J_2\,J_3$ of the two jet-chains in fig. 9. Just as in the motion considered above we have to build up the solution of the problem from phase to phase. If we assume $h$ to be zero, we might simply use the equations from the problem above introducing $\theta$ instead of $x$, $I$ instead of $m_0$ and so on. In the following examples we shall, however, also put $p = 0$ and then go back to the exact formulae based on (3). These formulae have in part been developed above on p. 13—14. A review of them corresponding to the first five phases is given in tab. I, the indications of which refer to fig. 10. The figures of the latter represent the same phases.

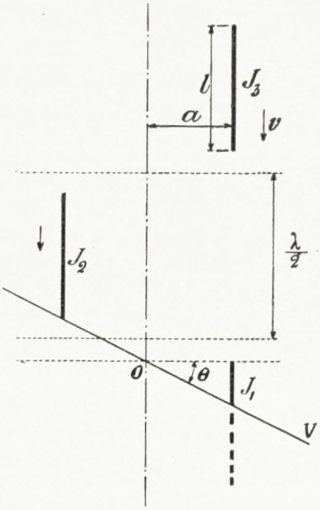

Fig. 9. See-Saw hit by two Jet-Chains.

By means of Tab. I in connection with fig. 10 we may study the motion in the case of $m = 1.70$ g/cm (diameter of jet 4 mm), $v = 600$ cm/sec, $l = 5$ cm, $\frac{\lambda}{2} = 6$ cm, $a = 3$ cm, $I = 460$ g/cm$^2$. With these constants $\frac{ma^2}{I} \cdot \frac{\lambda}{2} = 0.2$, and consequently the approximative formulae in tab. I may be applied. The result is reproduced in fig. 11 *A* where $\frac{T}{2} = \frac{\lambda/2}{v}$. As will be seen, the time elapsing from $J_1$ meets the bar to $J_3$ collides with it at the moment $t_4$ is only slightly greater than a period $T$ namely 1.06 $T$. But at the said moment the bar still possesses most of the positive deflection obtained. The period ending with only a very small negative angular velocity, it is obvious that the succeeding part of the curve must nearly be identical with the part already drawn. Furthermore in fig. 11 *B* a curve is drawn, which shows what the motion would be if we simply neglected the velocity of the see-saw $a\frac{d\theta}{dt}$, thus the damping force due to the jet itself. The motion is then represented by

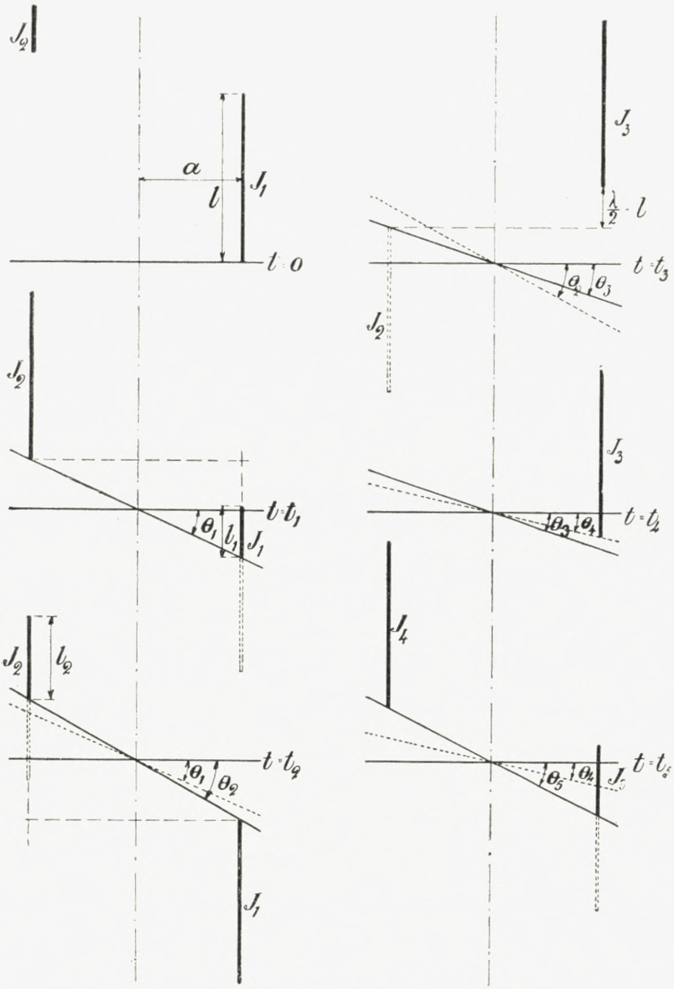


Fig. 10. Various Phases of the Motion of the See-Saw.

$$I \frac{d^2 \theta}{dt^2} = m a v^2 \tag{6}$$

thus by

$$\frac{d\theta}{dt} = \frac{m a v^2}{I} t + z_0 \tag{7}$$

$$\theta = \frac{1}{2} \frac{m a v^2}{I} t^2 + z_0 t + \theta_0 \tag{8}$$

$\theta_0$ and $z_0$ being the values of $\theta$ and $\frac{d\theta}{dt}$ at the moment $t = 0$. Obviously the motion has the same general character as the actual motion but the deflections are essentially larger, showing the great damping effect of the jet.

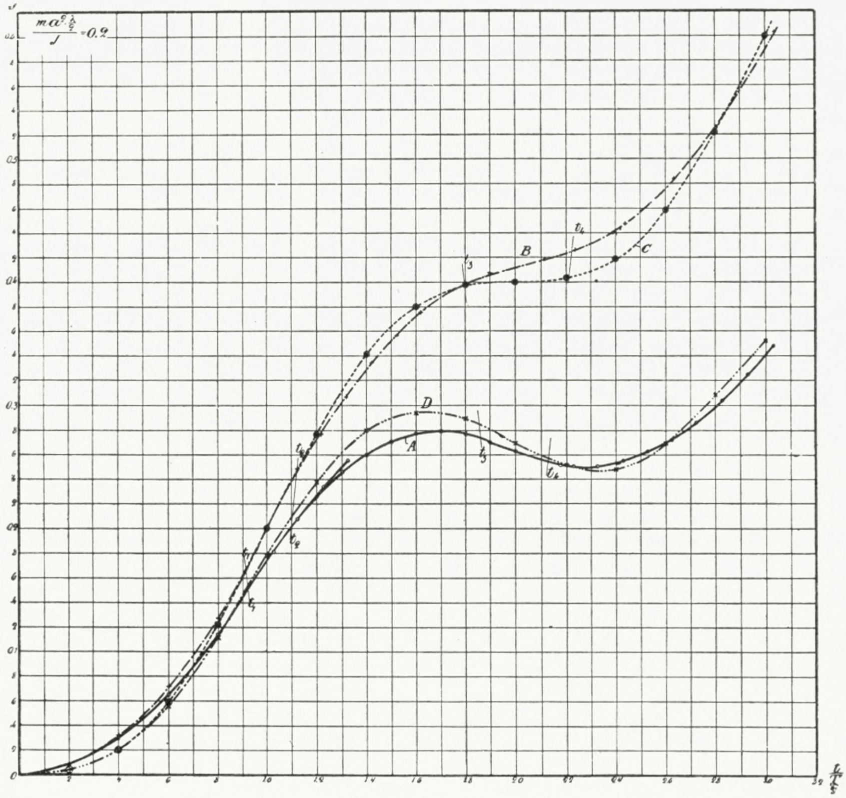


Fig. 11. Graphical Representation of the Motion for $\frac{ma^2\frac{\lambda}{2}}{I} = 0.2$.

Finally in fig. 11 $C$ the motion of the see-saw when acted on by a purely periodical moment

$$M = mv^2 a_0 \sin \omega t, \left(\omega = \frac{2\pi}{T}\right) \tag{9}$$

is represented. It has been assumed that $a_0 = \frac{\pi}{2} a$, $a$ being the arm in fig. 9. The motion is thus given by

$$I \frac{d^2\theta}{dt^2} = mv^2 a_0 \sin \omega t \tag{10}$$

## Tabel I.

| Phase | $\theta$ Complete Expressions |
|---|---|
| (1) | $\theta = \frac{vt}{a} - \frac{I}{ma^3} \log \text{nat} \left(1 + \frac{ma^2 \cdot vt}{I}\right)$ |
| (2) | $\theta = \theta_1 + z_1 \frac{I}{4\,ma^2 \cdot v} \left(1 - e^{-\frac{4\,ma^2 \cdot vt}{I}}\right)$ |
| (3) | $\theta = \theta_2 - \frac{vt}{a} + \frac{I}{ma^3} \log \text{nat} \left(1 + \frac{ma^3}{I} \left(\frac{v}{a} + z_2\right) \cdot t\right)$ |
| (4) | $\theta = \theta_3 + z_3 t$ |
| (5) | $\theta = \theta_4 + \frac{vt}{a} - \frac{I}{ma^3} \log \text{nat} \left(1 + \frac{ma^3}{I} \left(\frac{v}{a} - z_4\right) \cdot t\right)$ |
| | $\theta$ Approximate Expressions |
| (1) | $\theta = \frac{1}{2} \frac{ma^2 \lambda/2}{I} \cdot \frac{\lambda/2}{a} \cdot \left(\frac{t}{T/2}\right)^2 \left(1 - \frac{2}{3} \frac{ma^2 \lambda/2}{I} \left(\frac{t}{T/2}\right)\right)$ |
| (2) | $\theta = \theta_1 + z_1 t \left(1 - \frac{1}{2} \frac{4\,ma^2 \lambda/2}{I} \cdot \left(\frac{t}{T/2}\right)\right)$ |
| (3) | $\theta = \theta_2 + z_2 t - \frac{1}{2} \frac{ma^3}{I} \left(\frac{v}{a} + z_2\right)^2 t^2 \left(1 - \frac{2}{3} \frac{ma^3}{I} \left(\frac{v}{a} + z_2\right) t\right)$ |
| (4) | $\theta = \theta_3 + z_3 t$ |
| (5) | $\theta = \theta_4 + z_4 t + \frac{1}{2} \frac{ma^3}{I} \left(\frac{v}{a} - z_4\right)^2 t^2 \left(1 - \frac{2}{3} \frac{ma^3}{I} \left(\frac{v}{a} - z_4\right) t\right)$ |
| | $\frac{d\theta}{dt}$ |
| (1) | $\frac{d\theta}{dt} = \frac{v}{a} \cdot \frac{\frac{ma^2 \lambda/2}{I} \left(\frac{t}{T/2}\right)}{1 + \frac{ma^2 \lambda/2}{I} \left(\frac{t}{T/2}\right)}$ |
| (2) | $\frac{d\theta}{dt} = z_1 \cdot e^{-\frac{4\,ma^2 \cdot \lambda/2}{I} \cdot \frac{t}{T/2}} = z_1 \left(1 - \frac{4\,ma^2 \lambda/2}{I} \cdot \left(\frac{t}{T/2}\right)\right)$ |
| (3) | $\frac{d\theta}{dt} = -\frac{v}{a} + \left(\frac{v}{a} + z_2\right) \cdot \frac{1}{1 + \frac{ma^3}{I} \left(\frac{v}{a} + z_2\right) t}$ |
| (4) | $\frac{d\theta}{dt} = z_3$ |
| (5) | $\frac{d\theta}{dt} = \frac{v}{a} - \left(\frac{v}{a} - z_4\right) \frac{1}{1 + \frac{ma^3}{I} \left(\frac{v}{a} - z_4\right) t}$ |

Tabel I.

| Phase | $tg\,\theta$ | $l$ |
|---|---|---|
| (1) | $tg\,\theta_1 = \left(\frac{\lambda}{2} - vt_1\right) \cdot \frac{1}{a}$ | $l_1 = 2\,a\,tg\,\theta_1 + l - \frac{\lambda}{2}$ |
| (2) | $tg\,\theta_2 = \left(vt_2 + \frac{\lambda}{2} - l\right) \cdot \frac{1}{a} - tg\,\theta_1$ | $l_2 = \frac{\lambda}{2} - 2\,a\,tg\,\theta_2$ |
| (3) | $tg\,\theta_3 = \left(\frac{\lambda}{2} - vt_3\right) \cdot \frac{1}{a} - tg\,\theta_2$ | |
| (4) | $tg\,\theta_4 = \left(vt_4 - \left(\frac{\lambda}{2} - l\right)\right) \cdot \frac{1}{a} - tg\,\theta_3$ | |

the integral of which is

$$\theta = \frac{1}{2} \frac{ma^2 \lambda/2}{I} \cdot \frac{\lambda/2}{a} \left(\frac{t}{T/2} - \frac{1}{\pi} \sin\left(\pi \frac{t}{T/2}\right)\right). \tag{11}$$

As will be seen, the $C$-curve nearly coincides with the $B$-curve. From this we conclude that we might in the actual problem get a similarly good approximation by identifying the action of the two jet-chains with the action of a moment (9) combined with a damping

$$p = 2\,ma^2 v. \tag{12}$$

The equation of the motion would then be

$$I \frac{d^2\theta}{dt^2} + 2\,ma^2 v \cdot \frac{d\theta}{dt} = mv^2 a_0 \sin \omega t = M_0 \sin \omega t. \tag{13}$$

The solution of this equation is

$$\tag{14} \left\{ \begin{aligned} \theta = \frac{M_0}{\omega\sqrt{(I\omega)^2 + p^2}} \Bigg[ \sin\varphi + \frac{I\omega}{p} \cos\varphi \left(1 - e^{-\frac{p}{I\omega} \cdot 2\pi \cdot \frac{t}{T}}\right) \\ - \sin\left(2\pi \frac{t}{T} + \varphi\right) \Bigg] \end{aligned} \right.$$

where $tg\,\varphi = \dfrac{p}{I\omega}$.

By means of this formula the curve $D$ was computed. It shows a practically good agreement with $A$. Thus our conclusion is justified for the case considered. We shall preliminarily — as a hypothesis — assume that what has been found true for the first transient part of the motion will also hold good for the final stationary motion and that the motion of the see-saw vibrator may on the whole in most cases and also when an external damping $p$ and a directive moment $h$ is acting, be solved as a continuous problem by introducing the fictive moment (9) and the fictive damping (12). The problem has in this way been reduced to the ordinary problem of vibrations under the influence of a harmonic external force or moment. It should be noticed that the $a_0$ introduced in (9) is chosen in such a way that the actual arm $a$ is the mean value of $a_0 \sin \omega t \cdot \left(a = a_0 \cdot \frac{2}{\pi}\right)$.

In the example above the equation (14) becomes

$$\theta = 0.0630\left[7.92 - 7.79\, e^{-0.798 \cdot \frac{t}{T}} + \sin\left(2\pi\frac{t}{T} + 7^0\ 15'\right)\right].$$

In order to put our hypothesis to further test a new and rather extreme case was considered, namely

$$m = 1.70 \text{ g}, \quad v = 600 \text{ cm/sec}, \quad l = 6 \text{ cm}, \quad \frac{\lambda}{2} = 6 \text{ cm},$$
$$a = 3 \text{ cm}, \quad I = 18.36 \text{ g/cm}^2$$

thus a much lighter system, $\frac{ma^2\lambda/2}{I}$ being 5 against 0.2 in the former example. This change implies that the approximate formulae in tab. I can no longer be used but recourse must be had to the original formulae. It furthermore involves that the moments limiting the various phases cannot be found as the abscissae of the points of intersection between the $\theta$-curve and certain straight lines. It is necessary to replace the $\theta$-curve by the corresponding curve for $tg\theta$ in the way indicated in fig. 12 by the dotted curve-branches. It appears from the latter figure, in which the motion is represented by the $A$-curve that the damping due to the jets very soon after the lapse of the first phase checks the velocity and reduces it to naught. During the

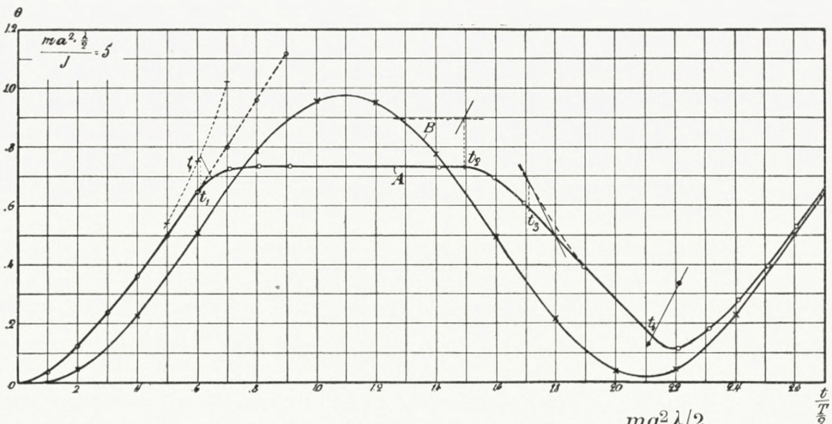

Fig. 12. Graphical Representation of the Motion for $\dfrac{ma^2\lambda/2}{I} = 5.$

rest of the second phase there is balance between the forces with which the two jet-pieces act on the see-saw. The third phase is very short but the jet-piece $J_2$ is nevertheless, during this phase, able to communicate a considerable velocity to the see-saw. With this velocity the latter moves on in the following fourth phase during which no jet-pieces act on the see-saw etc.

Applying now our hypothesis indicated above to the case of fig. 12, we find that the equation picturing the motion should be

$$\theta = 0.477 \cdot \left[1.048 - 0.094 \cdot e^{-20.1\frac{t}{T}} - \sin\left(2\pi\frac{t}{T} + 72^0\,36'\right)\right].$$

This relation is represented graphically in fig. 12$B$. Obviously the discrepancy between the two curves is now much more prominent than in the former case. The general character of the two curves $A$ and $B$ is, however, in some measure the same. We shall therefore, notwithstanding the divergences in such extreme cases as that just considered, assume that a fairly good approximation to the actual motion may generally be obtained by our method of substitution. We shall see in a following chapter that if the see-saw is hit by a jet-wave of sine-shape like that indicated in fig. 13a and if this wave is produced by a jet of velocity $v$ and mass per cm $m$ then the moment acting on the see-saw will just be

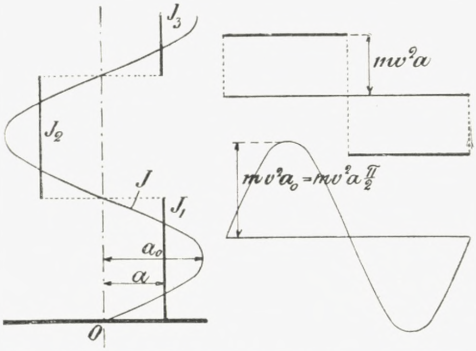


Fig. 13a—b. Nature of the Substitution.

$$(15)\qquad M = mv^2 a_0 \sin \omega t$$

provided the bar is kept in its normal position perpendicular to the axis of the wave. Under the same supposition the moment originating from the jet-chains $J_1 J_3 \ldots J_2 J_4 \ldots$ will be that indicated in the uppermost figure in fig. 13b. Our substitution is thus to the effect that we replace the jet-chain with the sine-wave when considering the motive moment, while we keep the jet-chain when considering the damping moment.

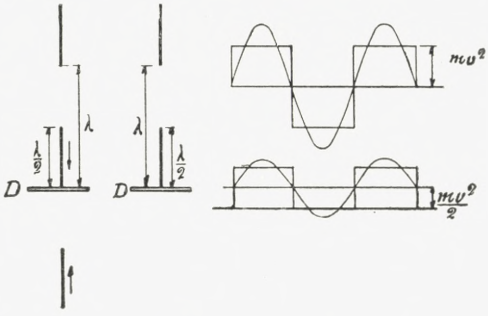


Fig. 14a—b. Substitution in Case of Translatory Vibrators.

Returning now for a moment to the jet-chain vibrator of the translatory type, there can be but little doubt that a substitution similar to that above may be applied in a good many cases. Thus with the twin-jet vibrator in fig. 14a the disk $D$ is, if kept in its zero position, acted on by a motive force like that represented in fig. 14a. We may probably substitute for this a harmonic force

(16) $$F = mv^2 \cdot \frac{\pi}{2} \sin \omega t$$

combined with a damping force $2\,mv \frac{dx}{dt}$. And in the case fig. 14b where the motive force may be considered as consisting of a constant force $\frac{mv^2}{2}$ and a rectangular alternating force of amplitude $\frac{mv^2}{2}$, the latter may probably be replaced by

(17) $$F = \frac{mv^2}{2} \cdot \frac{\pi}{2} \cdot \sin \omega t\,.$$

## 6. General Formulae of Motion of an oscillatory System.

Having reduced the problem of motion under the action of a jet-chain or a twin-jet-chain to the problem of an oscillatory system acted on by a simple harmonic force or moment, it seems appropriate to review the formulae governing the motion in the latter case.

Accordingly we consider the well known equation

(1) $$I \frac{d^2\theta}{dt} + p \frac{d\theta}{dt} + h\theta = M_0 \sin \omega t.$$

If (2) $$p^2 < 4Ih\,,$$

the complete solution is

(3) $$\begin{cases} \theta = e^{-\frac{p}{2I}t} \left[ a_1 \sin \frac{\sqrt{4Ih - p^2}}{2I} t + a_2 \cos \frac{\sqrt{4Ih - p^2}}{2I} t \right] \\ \qquad + \frac{M_0}{\sqrt{(h - I\omega^2)^2 + (p\omega)^2}} \sin (\omega t + \varphi) \end{cases}$$

where

$$tg\,\varphi = -\frac{p\,\omega}{h - I\omega^2} \tag{4}$$

and where $a_1$ and $a_2$ are constants to be determined by the values of $\theta$ and $\frac{d\theta}{dt}$ at a given moment say $t = 0$. We shall, however

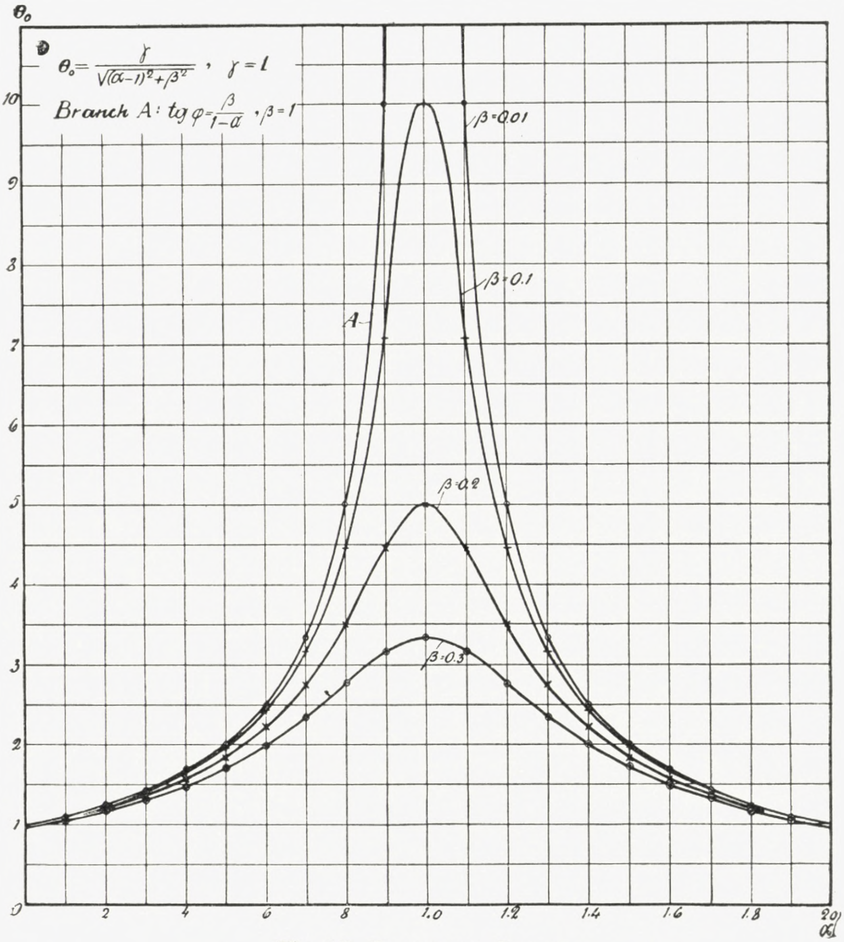


Fig. 15. Resonance-Curves.

here confine ourselves to the stationary part of the motion represented by the last term of (3) and characterized by the condition (2). Introducing the parameters

(5) $h = \alpha I \omega^2$, (6) $p\,\omega = \beta I \omega^2$, (7) $M_0 = \gamma \cdot I \omega^2$

we may write the expression for the amplitude $\theta_0$

(8) $$\theta_0 = \frac{\gamma}{\sqrt{(\alpha - 1)^2 + \beta^2}}$$

and

(9) $$tg\,\varphi = \frac{\beta}{1 - \alpha}.$$

In fig. 15 the function (8) has been drawn for $\gamma = 1$ and for four values of $\beta$, furthermore for the interval of $\alpha$ from $0 - 2$ which thus includes the case of resonance $\alpha = 1$. By means of the set of curves the absolute value of $\theta_0$ may be determined for any case within the $\alpha$ — and $\beta$ — intervals considered. One only has to multiply the value of $\theta_0$ taken from the set by the $\gamma$-value corresponding to the case in question. The branch $A$ very nearly determines $tg\,\varphi$ corresponding to $\beta = 1$ and so the value of $tg\,\varphi$ for any value of $\beta$. Of considerable interest is the question about the change of amplitude due to a change of frequency. In order to find the relation we shall have to differentiate the expression

(10) $$\theta_0 = \frac{M_0}{\sqrt{(h - I\omega^2)^2 + (p\,\omega)^2}}$$

with regard to $\omega$. The result may be written.

(11) $$\frac{\Delta\,\theta_0}{\theta_0} = \frac{2\,(\alpha - 1) + \beta^2}{(\alpha - 1)^2 + \beta^2} \cdot \frac{\Delta\,\omega}{\omega}.$$

In fig. 16 $\frac{\Delta\,\theta_0}{\theta_0} \Big/ \frac{\Delta\,\omega}{\omega}$ is represented in its relation to $\alpha$ and $\beta$. For instance we find from the curves that the ratio has the value $-2.8$ corresponding to $\alpha = 0.3$, $\beta = 0.1$. Thus an increase of 1 per cent of the frequency gives rise to a drop 2.8 per cent in the amplitude. The larger the damping the more independent of the frequency is the amplitude. — Almost as important is the relation between $\Delta\,tg\,\varphi$ and $\Delta\,\omega$. It is determined by differentiating

(12) $$tg\,\varphi = -\frac{p\,\omega}{h - I\omega^2}$$

and is found to be

(13) $$\Delta\,tg\,\varphi = -\frac{\beta}{1 - \alpha} \cdot \frac{1 + \alpha}{1 - \alpha} \cdot \frac{\Delta\,\omega}{\omega}.$$

The dotted curve in fig. 16 represents $\Delta\,tg\,\varphi \Big/ \frac{\Delta\,\omega}{\omega}$ corresponding to $\beta = 1$. If for instance $\alpha = 0.3$, $\beta = 0.1$ an increase of 1 per cent in $\omega$ will cause a decrease of $tg\,\varphi$ equal to $0.1 \cdot 2.7 \cdot \frac{1}{100} = 0.0027$ while $tg\,\varphi$ itself is seen from fig. 15 to be 0.144.

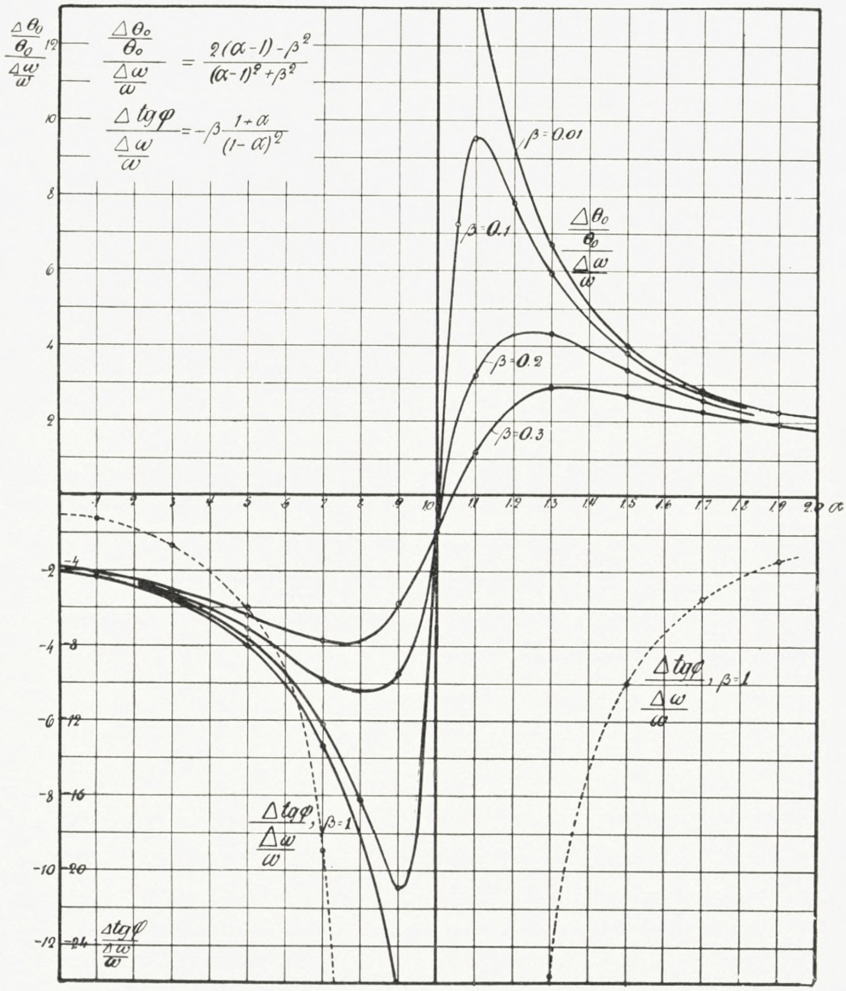


Fig. 16. Curves representing $\frac{\Delta\,\theta_0}{\theta_0}\Big/\frac{\Delta\,\omega}{\omega}$, $\Delta\, tg\,\varphi\Big/\frac{\Delta\,\omega}{\omega}$.

## 7. Some special Relations characteristic to the Jet-Chain See-Saw.

Returning to the special case of the see-saw we shall assume the latter, fig. 13a, to be acted on by a directive moment $h\theta$ and by a frictional moment $-p_0\frac{d\theta}{dt}$. Its differential equation then is

$$(1) \qquad I\frac{d^2\theta}{dt^2}+(p_0+2\,ma^2\,v)\frac{d\theta}{dt}+h\theta = mv^2\,a\frac{\pi}{2}\sin\omega t$$

and its motion, when it has become stationary, will be expressed by

$$(2) \qquad \begin{cases} \theta = \dfrac{mv^2\,a\dfrac{\pi}{2}}{\sqrt{(h-I\omega^2)^2+(p+2\,ma^2\,v)^2\,\omega^2}}\sin(\omega t+\varphi) \\ \quad = \theta_0\sin(\omega t+\varphi) \end{cases}$$

where

$$tg\,\varphi = -\frac{(p+2\,ma^2\,v)\cdot\omega}{h-I\omega^2}.$$

We shall in particular examine the amplitude and may by reason of the comparatively large effect of the damping due to the jet itself neglect the external damping. Then

$$(3) \quad \theta_0 = \frac{mv^2\,a\cdot\dfrac{\pi}{2}}{\sqrt{(h-I\omega^2)^2+4\,m^2\,a^4\,v^2\,\omega^2}} = \frac{\pi}{2}\cdot\frac{v}{\sqrt{\dfrac{A^2}{(mva)^2}+(2\,a\,\omega)^2}}.$$

From this expression, in which $A = h-I\omega^2$, it is seen that with a given oscillatory system the amplitude $\theta_0$ will vary practically proportional to $v$ when this latter quantity is raised above a certain limit. Furthermore, that with an increase of $m$, $\theta_0$ approaches a definite limit, namely

$$(4) \qquad \theta_{0,m} = \frac{1}{8}\,\frac{vT}{a} = \frac{1}{8}\,\frac{\lambda}{a}$$

and that finally there may be a certain value of $a$, which makes $\theta_0$ maximum. This value is in the ordinary way found to be determined by

$$(5) \qquad a_m = \frac{\sqrt{A}}{\sqrt{2\,\omega\,mv}}.$$

We may introduce the natural cyclic frequency $\omega'$ of

3*

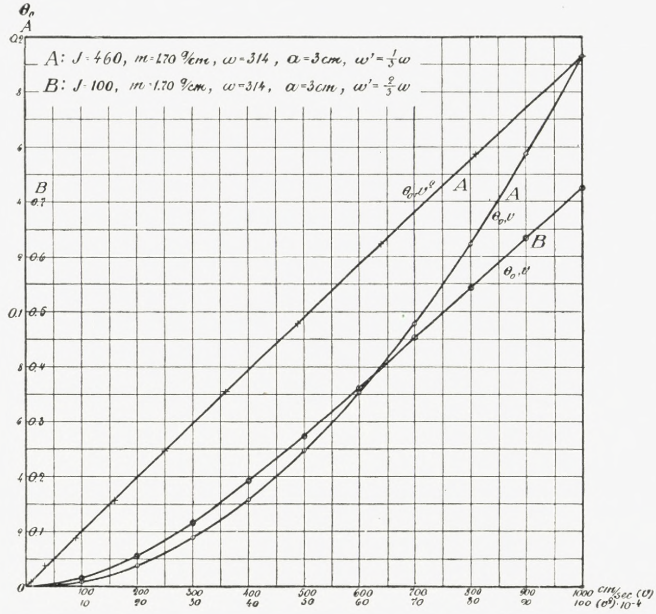


Fig. 17. $\theta_0 - v$, $\theta_0 - v^2$-Curves.

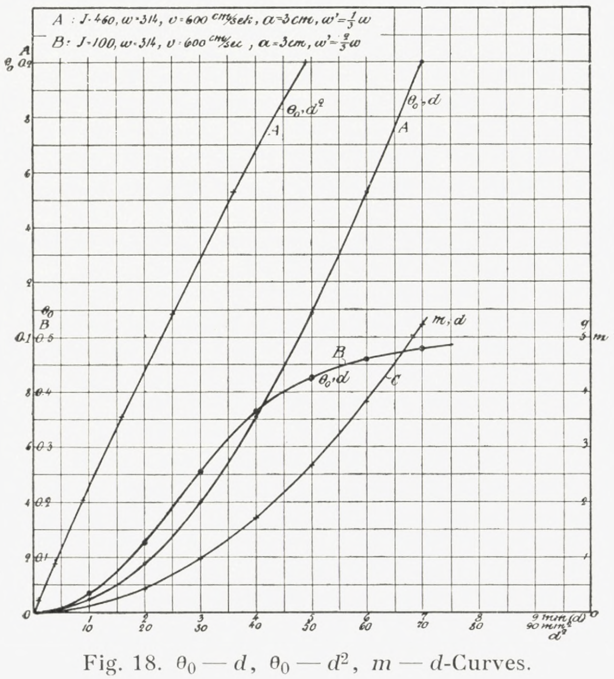


Fig. 18. $\theta_0 - d$, $\theta_0 - d^2$, $m - d$-Curves.

the oscillatory system. This quantity is determined by $\sqrt{\frac{h}{I}}$. We may thus write

$$a_m = \sqrt{I} \cdot \frac{\sqrt{(\omega'^2 - \omega^2)}}{\sqrt{2\,\omega\, m v}}. \tag{6}$$

We may finally write (3) as

$$\theta_0 = \frac{\pi}{2\,\omega} \cdot \frac{1}{\sqrt{\left(\frac{I\,\omega}{m v^2 a}\right)^2 \left(\left(\frac{\omega'}{\omega}\right)^2 - 1\right)^2 + \left(\frac{2\,a}{v}\right)^2}}. \tag{7}$$

If we introduce here the value (6) for $a$ we find the maximum of $\theta_0$

$$\theta_{0,m} = \frac{\pi}{2\,\omega} \cdot \frac{1}{\sqrt{\frac{4\,I\,\omega}{m v^3}\left[\left(\frac{\omega'}{\omega}\right)^2 - 1\right]}} \tag{8}$$

where the numerical value of the quantity $\left[\left(\frac{\omega'}{\omega}\right)^2 - 1\right]$ is to be used. From (7) we see that if $\omega' = \omega$, i. e. if there be resonance,

$$\theta_{0,m} = \frac{\pi}{2\,\omega} \cdot \frac{v}{2\,a} = \frac{1}{8} \cdot \frac{\lambda}{a}. \tag{9}$$

We thus get the same value of the amplitude as that to which the latter approaches when $m$- or $v$-increases beyond all measure, and what is highly interesting, a value quite independent of all parameters except $\lambda$ and $a$.

In order to illustrate the conditions indicated above, curves for the variation of the amplitude $\theta_0$ with $v$, $m$ and $a$ were calculated. They are reproduced in fig. 17—19 and as will be seen correspond to two rather different systems, one heavy with a frequency only $\frac{1}{3}$ of that of the turning moment acting on the system, and another comparatively light and with a natural frequency equal to

$\frac{2}{3}$ of $\omega$. In the first case $\theta_0$ is nearly proportional to $v^2$, $m$ and to $a$. In the latter case the damping member $\left(\frac{2\,a}{v}\right)^2$ in (7) plays a considerable part and we therefore see that the

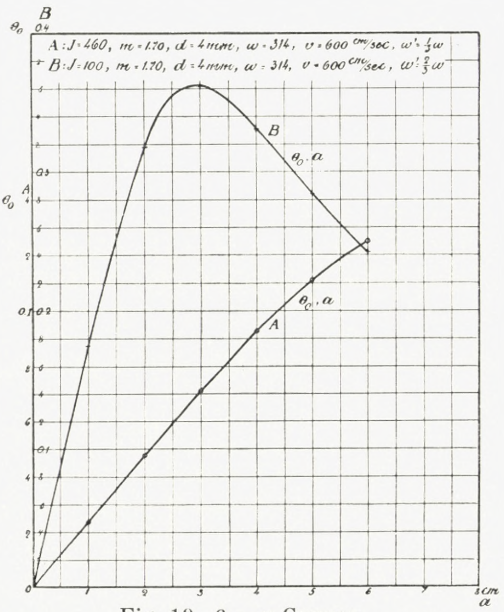


Fig. 19. $\theta_0$—$a$-Curves.

$\theta_0$—$v$-curve for greater values of $v$ approaches a straight line, while the $\theta_0$—$d$- and so the $\theta_0$—$m$-curve approaches the constant value 0.5 and the $\theta$—$a$-curve exhibits a very pronounced maximum.

II.

# The Jet-Wave-Vibrator.

## 1. The Vibrator with a Jet-Wave of constant Amplitude.

We will consider the motion of a balance or see-saw $B$, fig. 20, hit by a jet-wave $J$ of constant amplitude $a_0$. We assume the axis of the wave to pass through that of the balance 0 and to be perpendicular to the latter. We shall first derive an expression for the moment with which the jet-wave acts on the see-saw, when the same is kept in its normal position perpendicular to the axis of the wave.

The particle $ds$ of the wave contains the same mass of liquid as the element $dx$ of the jet from which the wave is made, $dx$ being the projection of $ds$ on the axis of the wave. The mass referred to is thus $m \cdot dx$ if $m$ denotes the mass per cm of the original jet. It moves down against the bar of the see-saw with the velocity $v$ of the original jet, carrying with it the momentum $mv\,dx$. In the course of the time $\frac{dx}{v}$ all the momentum perpendicular to $B$ is destroyed. It means that the bar is acted on by a force

$$(1) \qquad F = \frac{mv \cdot dx}{(dx/v)} = mv^2 .$$

The corresponding turning moment is

$$(2) \qquad M = mv^2 a .$$

Now $a$ varies with time according to

$$a = a_0 \sin \frac{2\pi}{T} t = a_0 \sin \omega t. \tag{3}$$

Thus

$$M = mv^2 a_0 \sin \omega t. \tag{4}$$

We will now suppose that the bar is kept in a position forming the angle $\theta$ with the normal position. Again the component of the momentum perpendicular to the deflected bar is destroyed during the collision, thus $mv \cos \theta \cdot dx$. But the time required for the destruction now greatly depends on the element, as will be seen from fig. 20a. The element $ds$ will not have "passed" the bar completely until the moment when $a$ reaches $b$. It means that the collision takes the time

$$dt = \frac{dx - dy \cdot tg\,\theta}{v} = \left(1 - tg\,\theta \frac{dy}{dx}\right) \cdot \frac{dx}{v}. \tag{5}$$

The force perpendicular to the bar thus becomes

$$F_n = mv^2 \cdot \cos \theta \cdot \frac{1}{1 - tg\,\theta \cdot \frac{dy}{dx}}. \tag{6}$$

Now the equation of the jet-wave at a given moment $t$ may be written

$$y = a_0 \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \tag{7}$$

from which

$$\frac{dy}{dx} = -a_0 \cdot \frac{2\pi}{\lambda} \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) = -\frac{2\pi}{\lambda} \cdot \sqrt{a_0^2 - y^2}. \tag{8}$$

Introducing this in (6) we get

$$F_n = mv^2 \cos \theta \cdot \frac{1}{1 + \frac{2\pi}{\lambda} tg\,\theta \cdot \sqrt{a_0^2 - y^2}}. \tag{9}$$

Now the turning moment $M$ is $F_n \cdot a' = F_n\, y/\cos\theta$, thus

(10) $$M = mv^2 \lambda \cdot \frac{y/\lambda}{1 + \frac{2\pi}{\lambda}\, tg\,\theta \sqrt{a_0^2 - y^2}}.$$

Finally we may take the zero-point of the system of co-

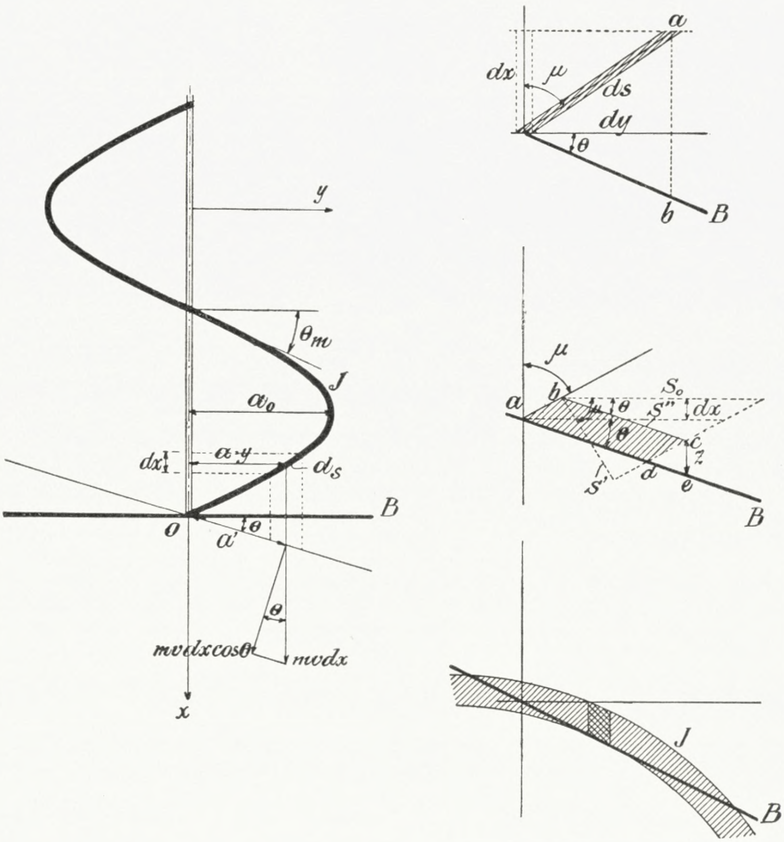


Fig. 20 a

Fig. 20 b

Fig. 20 c

Fig. 20. See-saw hit by a Jet-Wave of constant Amplitude.

ordinates to coincide with $O$, fig. 20. Then $x = y\, tg\,\theta$ and from (7) we get

(11) $$y = a_0 \sin 2\pi \left( \frac{t}{T} - y \frac{tg\,\theta}{\lambda} \right).$$

If we want to express $M$ as a function of the time $t$, we must solve (11) with regard to $y$ and afterward introduce the result in (10). The practical way to produce a picture of the relation between $M$ and $t$ is first to calculate, by means of (11), $t$ corresponding to a number of values of $y$. The result may be represented in the shape of a curve showing $\frac{y}{a_0}$ as a function of $\frac{t}{T}$. Furthermore a curve must be pro-

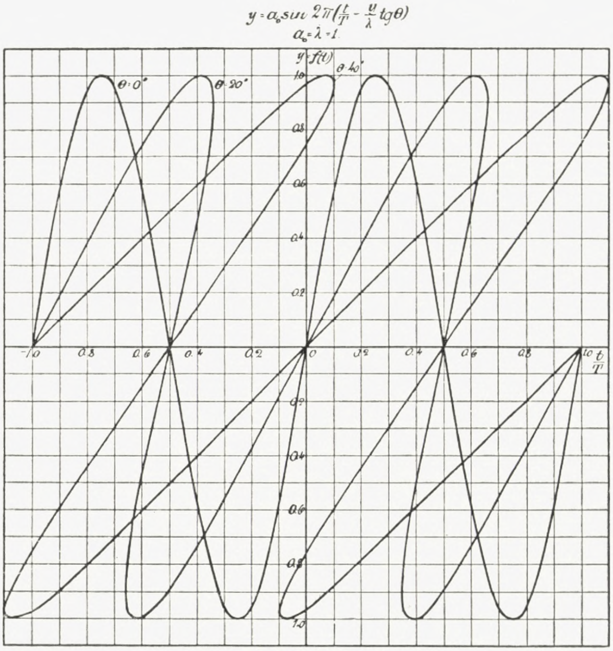


Fig. 21. Curves representing $y = a_0 \sin 2\pi \left(\frac{t}{T} - \frac{y}{\lambda} tg\,\theta\right)$.

duced representing $M$ as a function of $y$, thus the graphical picture of (10). Now combining the two curves, a picture of $M$ as a function of $t$ is easily obtained.

We shall here confine ourselves to illustrating what has been said above by means of the curves in fig. 21, representing the function (11) for three values of $\theta$ and for $a_0 = \lambda = 1$. The curve corresponding to $\theta = 0$ is of course the simple sine-curve. The curves for $\theta = 20°$ and $\theta = 40°$ exhibit a number of values of $y$ corresponding to the same value of $t$. The explanation is very simple. For if the deflection of the balance $B$ in fig. 20 is sufficiently great i. e. greater than $\theta_m$, then the bar is cut simultaneously in several places by the wave. It may be noticed that $\theta_m$ is determined by

$$tg\,\theta_m = \frac{\lambda}{2\pi a_0} \tag{12}$$

or if $\lambda = a_0$, $\theta_m = 9^\circ 3'$.

The expression (6) may be obtained in a somewhat different way as indicated in fig. 20b. The element of the jet-wave is here confined between two planes parallel to the surface of $B$. The momentum of the element is, with the indications of the figure, $v \varrho S'' z \cos\theta$ ($\varrho$ the density of the liquid). The particle disappears into $B$ in the course of the time $\frac{z}{v}$. Thus the force perpendicular to $B$ is $\varrho S'' v^2 \cos^2\theta$. Now it appears from the figure that $S'' = S'/\cos(\mu - \theta)$ if $S'$ is the area of the normal cut through the wave-element. Furthermore $S' = S_0 \cos\mu$, $S_0$ being the cross-section of the original jet. Thus

$$\tag{13} \left\{ \begin{aligned} F_n &= \varrho S_0 \frac{\cos\mu}{\cos(\mu - \theta)} v^2 \cos^2\theta \\ &= m v^2 \cos\theta \cdot \frac{\cos\theta\cos\mu}{\cos(\mu - \theta)} \\ &= m v^2 \cos\theta \cdot \frac{1}{1 + tg\,\mu\, tg\,\theta} \end{aligned} \right.$$

which expression is identical with (6), $tg\,\mu$ being equal to $-\frac{dy}{dx}$.

It may be remarked that our method of calculating the force and turning moment fails if

$$\frac{dy}{dx} = \cot\theta \tag{14}$$

for in that case the expression (6) gives $F_n = \infty$. The case referred to is that indicated in fig. 20c, i. e. the bar has become a tangent to the wave-element. Of course the force or moment is not infinite. In order to obtain its actual value we only have to employ another method of dividing the wave into elements, thus for instance that indicated by the double hatching. We shall not, however, bother about the problem, it being practically of very little interest, even though we shall meet with it in some of the examples presented in the following.

In case of the deflection $\theta$ being so small that we may put $\cos 2\pi \frac{y\,tg\,\theta}{\lambda} = 1$, the expression (11) may be replaced by

$$(15) \qquad y = a_0 \sin 2\pi \frac{t}{T} - a_0 \cdot \frac{2\pi y\, tg\,\theta}{\lambda} \cdot \cos 2\pi \frac{t}{T}$$

from which we find

$$(16) \quad \left\{ \begin{aligned} y &= \frac{a_0 \sin 2\pi \frac{t}{T}}{1 + \frac{2\pi\, tg\,\theta}{\lambda} a_0 \cos 2\pi \frac{t}{T}} \\ &= a_0 \sin 2\pi \frac{t}{T} \left(1 - \frac{2\pi\, tg\,\theta}{\lambda} a_0 \cos 2\pi \frac{t}{T}\right). \end{aligned} \right.$$

Introducing in (10) after having replaced $\sqrt{a_0^2 - y^2}$ by $a_0 \cos \frac{2\pi}{T} t$ we get

$$(17) \qquad M = mv^2 a_0 \sin 2\pi \frac{t}{T} \left(1 - 4\pi \frac{a_0}{\lambda}\, tg\,\theta \cdot \sin \frac{2\pi}{T} t\right).$$

If finally $4\pi \frac{a_0}{\lambda}\, tg\,\theta$ is small compared to 1

$$(17) \qquad M = mv^2 a_0 \sin \frac{2\pi}{T} \cdot t$$

i. e. in the first approximation the moment is independent of $\theta$.

We may now write down the differential equation for the motion of the see-saw in the latter case. In so doing we shall replace $v$ in (17) by the relative velocity $v_r$ which is determined by

$$(18) \qquad v_r = v - y \cdot \frac{d\theta}{dt} = v - a_0 \sin \omega t \cdot \frac{d\theta}{dt}.$$

Thus

$$(19)\quad \left\{\begin{aligned} M &= ma_0\left(v - a_0 \sin \omega t \cdot \frac{d\theta}{dt}\right)^2 \cdot \sin \omega t \\ &= mv^2 a_0 \sin \omega t - 2\, ma_0^2 v \sin^2 \omega t \cdot \frac{d\theta}{dt}. \end{aligned}\right.$$

And the equation in question becomes

$$(20)\quad I\frac{d^2\theta}{dt^2} + (p + 2\, ma_0^2\, v \sin^2 \omega t)\,\frac{d\theta}{dt} + h\theta = mv^2 a_0 \sin \omega t$$

provided the see-saw has the character of an oscillatory system with a moment of inertia $I$, an external damping $p\dfrac{d\theta}{dt}$ and a directive moment $h\theta$. If $p = 0$ we may write

$$(21)\quad I\frac{d^2\theta}{dt^2} + ma_0^2 v \cdot (1 - \cos 2\,\omega t)\cdot\frac{d\theta}{dt} + h\theta = mv^2 a_0 \sin \omega t.$$

The coefficient of $\dfrac{d\theta}{dt}$, i. e. the damping-factor, thus varies with time. Its mean value is $ma_0^2 v$. Probably it will prove difficult to solve the complete equations (20) or (21) but it seems likely that we shall obtain a fairly good first approximation by simply neglecting the variations of the damping, i. e. by solving, in the case of $p = 0$, the equation

$$(22)\qquad I\frac{d^2\theta}{dt} + ma_0^2\, v\frac{d\theta}{dt} + h\theta = mv^2 a_0 \sin \omega t.$$

The problem has thus been reduced to the ordinary problem of the motion of an oscillatory system under the influence of a harmonic moment.

It is interesting to compare the present system with the oscillatory system acted on by a twin jet-chain. In the latter case we substituted for the jet-chain a jet-wave, the amplitude $a_0$ of which was $\dfrac{\pi}{2}$ times the arm $a$ on which the chain acted. At the same time we kept the damping factor $2\, ma^2 v$, of the jet-chain action. Now, when the oscillatory system is actually acted on by a jet-wave we keep its motive force, $mv^2 a_0 \sin \omega t$, but replace its damping by the damping of a twin jet-chain, the arm of which is $a = \frac{a_0}{\sqrt{2}}$ $\left(\text{not } a_0 \cdot \frac{2}{\pi}\right.$, which might have been expected and which is not very different from $\left.\frac{a_0}{\sqrt{2}}\right)$. On the whole the problems of the two kinds of motions have been reduced to one. Thus it will not here be necessary to repeat the discussion with respect to the amplitude which has already been given in the treatment of the jet-chain-vibrator.

## 2. The Turning Moment in the Case of a Jet-Wave of the circular Type.

We shall next consider the see-saw hit by a jet-wave of the circular type i. e. the wave which may be produced by the interaction of a constant magnetic field and an alternating current passed through a conductive liquid-jet. The characteristic property of this wave may be said to be that an element $ds$, fig. 22, of the wave contains the same mass of liquid as does its circular projection $dr$ of the original jet. The equation of the jet-wave at a given moment $t$ may with polar coordinates $r$, $\alpha$ be written

$$(1) \qquad \sin \alpha = \sin \alpha_0 \sin\left(\omega t - \omega \frac{r}{v}\right).$$

On the basis of similar considerations as in the case of the wave with constant amplitude we derive for the force perpendicular to the deflected bar the expression

$$(2) \qquad F_n = mv^2 \cos (\theta + \alpha) \cdot \frac{dr}{dr - dz}$$

$\frac{-dr + dz}{v}$ being the time which it takes for the element $ds$ to "pass" the bar. The turning moment corresponding to $F_n$ is

(3) $$M = F_n \cdot a' = F_n \cdot a \frac{\cos\alpha}{\cos(\alpha+\theta)} = F_n \cdot \frac{r_0 \sin\alpha}{\cos(\alpha+\theta)}$$

thus

(4) $$M = mv^2 r_0 \sin\alpha \cdot \frac{dr}{dr - dz}.$$

It is now seen from fig. 22 that

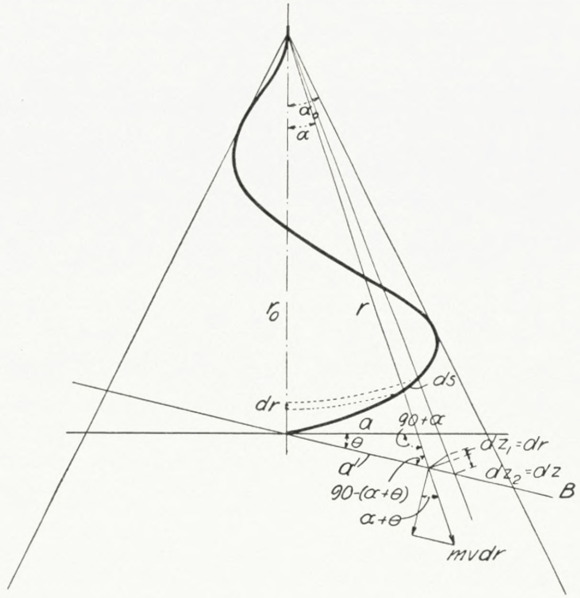


Fig. 22. See-Saw hit by a Jet-Wave of circular Type.

(5) $$r = r_0 \frac{\cos\theta}{\cos(\alpha+\theta)}$$

from which

(6) $$\frac{dr}{d\alpha} = \frac{dz}{d\alpha} = r_0 \cos\theta \cdot \frac{\sin(\alpha+\theta)}{\cos^2(\alpha+\theta)}.$$

Introducing in (4) we get

(7) $$M = mv^2 r_0 \sin\alpha \cdot \frac{1}{1 + r_0 \dfrac{\cos\theta \, \sin(\alpha+\theta)}{\cos^2(\alpha+\theta)} \cdot \dfrac{d\alpha}{dr}}.$$

The value of $\frac{d\alpha}{dr}$ is found from (1):

$$(8)\quad \frac{d\alpha}{dr} = -\frac{\sin\alpha_0 \cos\left(\omega t - \omega \frac{r}{v}\right)}{\cos\alpha} \cdot \frac{\omega}{v} = -\frac{\sqrt{\sin^2\alpha_0 - \sin^2\alpha}}{\cos\alpha} \cdot \frac{\omega}{v}.$$

Thus

$$(9)\quad M = mv^2 r_0 \sin\alpha \frac{1}{1 + 2\pi \frac{r_0}{\lambda}\cos\theta \frac{\sin(\alpha+\theta)\sqrt{\sin^2\alpha_0 - \sin^2\alpha}}{\cos^2(\alpha+\theta)\cos\alpha}}.$$

If $\alpha$ and $\theta$ are both so small that we may put $\cos\alpha$, $\cos\theta$, and $\cos(\alpha+\theta) = 1$ and if $\frac{r_0}{\lambda}\cdot 2\pi$ is not too great, the expression (9) may be written

$$(10)\quad \left\{\begin{aligned} M &= mv^2 r_0 \sin\alpha_0 \sin\left(\omega t - \omega\frac{r}{v}\right) \\ &\quad \cdot\left[1 - \frac{r_0}{\lambda}\cdot 2\pi \sin(\alpha+\theta)\sin\alpha_0 \cos\left(\omega t - \omega\frac{r}{v}\right)\right] \\ &= mv^2 r_0 \sin\alpha_0 \sin\left(\omega t - \omega\frac{r}{v}\right) \\ &\quad \cdot\left[1 - 2\pi\frac{r_0}{\lambda}\sin^2\alpha_0 \sin\left(\omega t - \omega\frac{r}{v}\right)\cos\left(\omega t - \omega\frac{r}{v}\right)\right. \\ &\quad \left. - 2\pi\frac{r_0}{\lambda}\sin\theta\sin\alpha_0\cos\left(\omega t - \omega\frac{r}{v}\right)\right]. \end{aligned}\right.$$

Now, if in the general case we want to find the moment $M$ as a function of the time $t$, we should first introduce the $r$ taken from (5) in the expression (1), thus obtaining the equation

$$(11)\quad \sin\alpha = \sin\alpha_0 \sin\left(\omega t - \frac{\omega r_0}{v}\cdot\frac{\cos\theta}{\cos(\alpha+\theta)}\right).$$

The latter equation should then be solved with regard to $\alpha$ and the result introduced in (9). Actually the problem must of course be treated in the manner indicated above. I. e. first we shall ascribe a number of values to $\alpha$ in (11) and find the corresponding values of $t$, afterwards representing $\alpha$ graphically as a function of $t$. Next we shall by means of (9) produce a graph of $M$ as a function of $\alpha$. Finally, combining the two graphs, we may draw the curve representing the relation between $M$ and $t$ corresponding to the angle $\theta$ of the deflection in question.

We shall now illustrate the production of the $M$—$t$-curve by an example. We choose the case $\theta = 20^\circ$, $\alpha_0 = 0.5$,

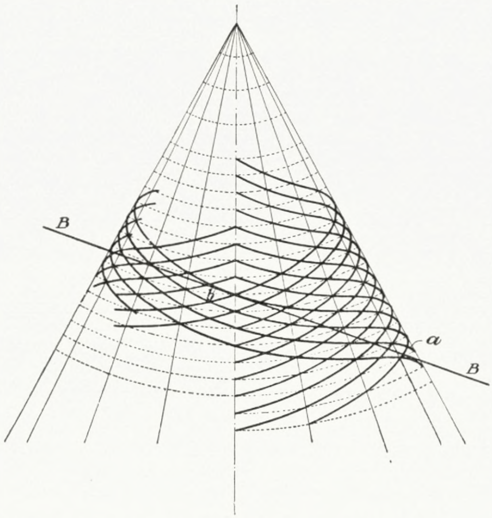


Fig. 23. Jet-Wave in a Series of Positions.

$\frac{r_0}{\lambda} = 1$ from which $\frac{\omega r_0}{v} = 2\pi \frac{r_0}{\lambda} = 2\pi$. In fig. 23 the wave has been drawn in a series of positions. In the picture, the construction of which we shall not here explain[1]), the deflected bar of the see-saw is indicated as $B\ B$. Tab. I first shows the numerical determination of the relation between $\alpha$ and $t$ or rather $\frac{t}{T}$ given by (11). The two solutions of (11)

$$(12)\quad \left\{ 2\pi\left(\frac{t}{T} - \frac{\cos\theta}{\cos(\alpha+\theta)}\right) \right\} = \left\{ \begin{array}{l} \arcsin\left(\frac{\sin\alpha}{\sin\alpha_0}\right) \\ \pi - \arcsin\left(\frac{\sin\alpha}{\sin\alpha_0}\right) \end{array} \right.$$

[1]) Compare: The Jet-Wave. Vidensk. Selsk. Math.-fys. Medd. IX, 2. p. 35.

obviously correspond to the two intervals $0—\alpha_0$ and $\alpha_0—0$. Tab. I is represented graphically in fig. 24. The curve exhibits up to three values of $\alpha$ corresponding to certain values of $t$. This will be understood from fig. 23 which shows that the bar with certain positions of the jet-wave

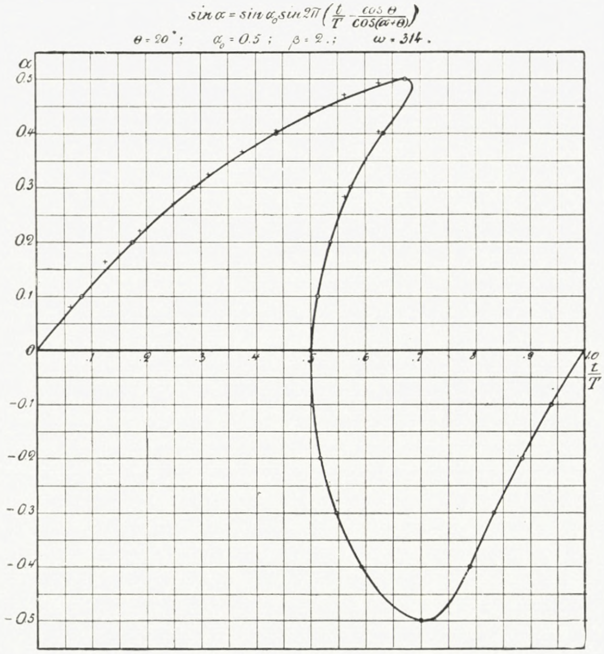


Fig. 24. $\alpha$—$t$-Curve, $\alpha_0 = 0{,}5$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

cuts the latter in three points. It may furthermore be noted that the construction in fig. 23 may be used for a direct determination of the curve in fig. 24. The waves drawn correspond to positions equidistant with regard to time. Thus if we draw the radii to the points of intersection between *BB* and the wave-pictures and measure the $\alpha$'s of the radii, we have $\alpha$ as a function of time. In this, not very accurate, way the points in fig. 24 marked by crosses are determined.

The next part of our problem consists in the numerical and graphical determination of the $M$—$\alpha$-relation given by (9) and originating from (4). In the calculation a certain account must be kept of the sign of the members in the denominator of (9). The simplest way to do this is probably

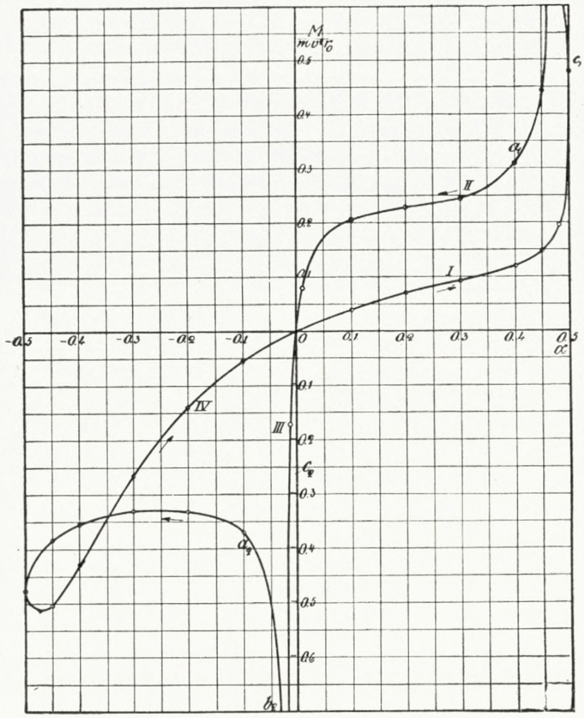


Fig. 25. $M$—$\alpha$-Curve, $\alpha_0 = 0.5$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

to inspect a construction like fig. 23. In tab. II some of the results of the numerical determination have been recorded and in fig. 25 the complete $M$—$\alpha$-curve is drawn. There is one remarkable thing about the curve, namely that, corresponding to two values of $\alpha$, it exhibits infinite values of $M$. In order to understand this we shall again have to inspect fig. 23. From this figure it is seen that, corresponding to two values of $\alpha$, the jet-wave will touch $B\,B$. (The points $a$ and $b$). When this occurs we have seen that our theory of the mechanical force and of the turning moment fails. Thus the two points of the curve in fig. 25 are not to be taken seriously. However, we shall not try to determine the curve-branches by which they ought to be re-

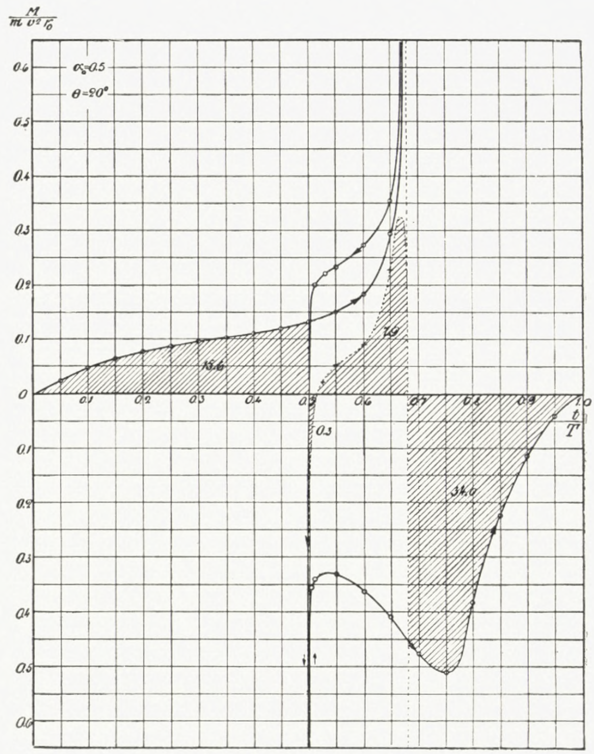

Fig. 26. $M$—$t$-Curve, $\alpha_0 = 0.5$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

placed, these singularities playing only a small part in the final result.

This is reproduced in fig. 26, which thus represents $M$ as a function of $t$. Obviously $M$ may at the same time exhibit up to three values, actually, however, the three values combine to one which is obtained by simply adding the components. This has been done and the re-

Table I.

| (1) $\alpha$ | (2) $\alpha$ | (3) $\sin\alpha$ | (4) $\frac{\sin\alpha}{\sin\alpha_0}$ | (5) $2\pi\left(\frac{t}{T}\right.$ | (6) $\left.-\frac{\cos\theta}{\cos(\alpha+\theta)}\right)$ |
|---|---|---|---|---|---|
| 0 | 0° | 0.0000 | 0.000 | 0° | 0.000 |
| 0.1 | 5°44′ | 0.0998 | 0.208 | 12° 3′ | 0.210 |
| 0.2 | 11°27′ | 0.1985 | 0.414 | 24°27′ | 0.427 |
| 0.3 | 17°12′ | 0.2957 | 0.616 | 38° 3′ | 0.664 |
| 0.4 | 22°54′ | 0.3891 | 0.812 | 54°18′ | 0.948 |
| 0.5 | 28°36′ | 0.4787 | 1.000 | 90° | 1.571 |

| (1) $\alpha$ | (7) $\cos(\alpha+\theta)$ | (8) $\frac{\cos\theta}{\cos(\alpha+\theta)}$ | (9) $\frac{6°}{2\pi}$ | (10) $\frac{t}{T}$ | (11) 0.5 — (9) | (12) $\frac{t}{T}$ |
|---|---|---|---|---|---|---|
| 0 | 0.940 | 1.000 | 0.0000 | 1.000 | 0.5000 | 1.500 |
| 0.1 | 0.900 | 1.048 | 0.0334 | 1.081 | 0.4666 | 1.515 |
| 0.2 | 0.853 | 1.105 | 0.0680 | 1.173 | 0.4320 | 1.537 |
| 0.3 | 0.797 | 1.181 | 0.1059 | 1.287 | 0.3941 | 1.575 |
| 0.4 | 0.733 | 1.285 | 0.1511 | 1.436 | 0.3489 | 1.634 |
| 0.5 | 0.661 | 1.422 | 0.2502 | 1.672 | 0.2498 | 1.672 |
| 0 | 0.940 | 1.000 | 0.0000 | 1.000 | 0.5000 | 1.500 |
| — 0.1 | 0.969 | 0.971 | — 0.0334 | 0.938 | 0.5334 | 1.504 |
| — 0.2 | 0.989 | 0.951 | — 0.0680 | 0.883 | 0.5680 | 1.519 |
| — 0.3 | 0.999 | 0.940 | — 0.1059 | 0.834 | 0.6059 | 1.546 |
| — 0.4 | 0.999 | 0.940 | — 0.1511 | 0.789 | 0.6511 | 1.591 |
| — 0.5 | 0.989 | 0.950 | — 0.2502 | 0.700 | 0.7502 | 1.700 |

sultant $M$—$t$-curve in fig. 26 is that which limits the hatched area.

We shall now explain our reason for spending a good deal of time on the problem of producing the $M$—$t$-curve. It was done in order to find the cause of a characteristic property of the see-saw hit by a circular jet wave. Preliminary observations showed that the see-saw without any

# Table II.

$$2\pi \cos \theta = 5.90$$

$$L = 5.90 \cdot \frac{\sin(\alpha+\theta)\sqrt{\sin^2\alpha_0 - \sin^2\alpha}}{\cos^2(\alpha+\theta)\cos\alpha}$$

| $\alpha$ | $\sin\alpha$ | $\cos\alpha$ | $\sin(\alpha+\theta)$ | $\cos(\alpha+\theta)$ | $\sqrt{\sin^2\alpha_0 - \sin^2\alpha}$ | $L$ | $\overline{M}_1 = \frac{\sin\alpha}{1+L}$ | $\overline{M}_2 = \frac{\sin\alpha}{-1+L}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.0000 | 1.0000 | 0.342 | 0.940 | 0.479 | 1.095 | 0.000 | 0.000 |
| 0.1 | 0.0998 | 0.9950 | 0.434 | 0.900 | 0.469 | 1.488 | 0.040 | 0.205 |
| 0.2 | 0.1985 | 0.9801 | 0.522 | 0.853 | 0.436 | 1.880 | 0.069 | 0.226 |
| 0.3 | 0.2957 | 0.9553 | 0.605 | 0.797 | 0.377 | 2.220 | 0.092 | 0.242 |
| 0.4 | 0.3891 | 0.9212 | 0.681 | 0.733 | 0.280 | 2.270 | 0.119 | 0.306 |
| 0.5 | 0.4787 | 0.8780 | 0.750 | 0.661 | 0.000 | 0 | 0.479 | 0.000 |
| | | | | | | | $\overline{M}_4 = \frac{\sin\alpha}{1+L}$ | $\overline{M}_3 = \frac{\sin\alpha}{1-L}$ |
| 0 | 0.0000 | 1.0000 | 0.342 | 0.940 | 0.479 | 1.095 | 0.000 | 0.000 |
| — 0.1 | — 0.0998 | 0.9950 | 0.246 | 0.969 | 0.469 | 0.730 | — 0.058 | — 0.370 |
| — 0.2 | — 0.1985 | 0.9801 | 0.149 | 0.989 | 0.436 | 0.400 | — 0.142 | — 0.331 |
| — 0.3 | — 0.2957 | 0.9553 | 0.049 | 0.999 | 0.377 | 0.114 | — 0.265 | — 0.333 |
| — 0.4 | — 0.3891 | 0.9212 | — 0.051 | 0.999 | 0.280 | — 0.092 | — 0.429 | — 0.356 |
| — 0.5 | — 0.4787 | 0.8780 | — 0.150 | 0.989 | 0.000 | — 0.000 | — 0.479 | — 0.479 |

directive moment will always oscillate about a position perpendicular to the axis of the wave. Thus the system has in a way a directive moment of its own. Now the hatched areas over the $\frac{t}{T}$ axis in fig. 26 represent the time

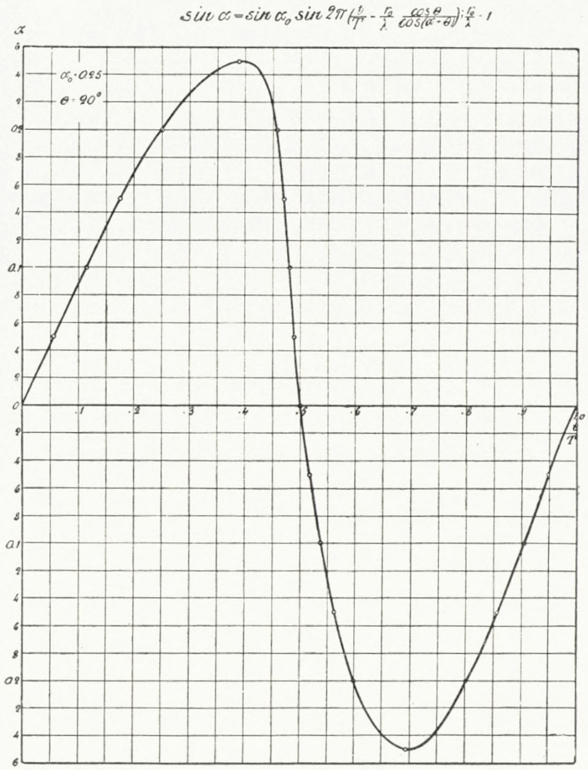

Fig. 27. $\alpha$—$t$-Curve, $\alpha_0 = 0.25$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

integral of the moment which will increase the deflection $\theta$, while the area under the axis corresponds to the moment which will reduce the deflection. The areas were measured by means of a planimeter. The results are written on the areas, and it is seen that a time-integral of ab. 34.3 tends to reduce $\theta$ while only a total time integral of 23.5 tends to increase $\theta$. We thus understand the tendency of the bar to return from a position of deflection to the position perpendicular to the axis of the jet-wave.

This tendency, however, is only pronounced if both $\alpha_0$ and the deflection $\theta$ are not too small. Thus if $\alpha_0 = 0.25$,

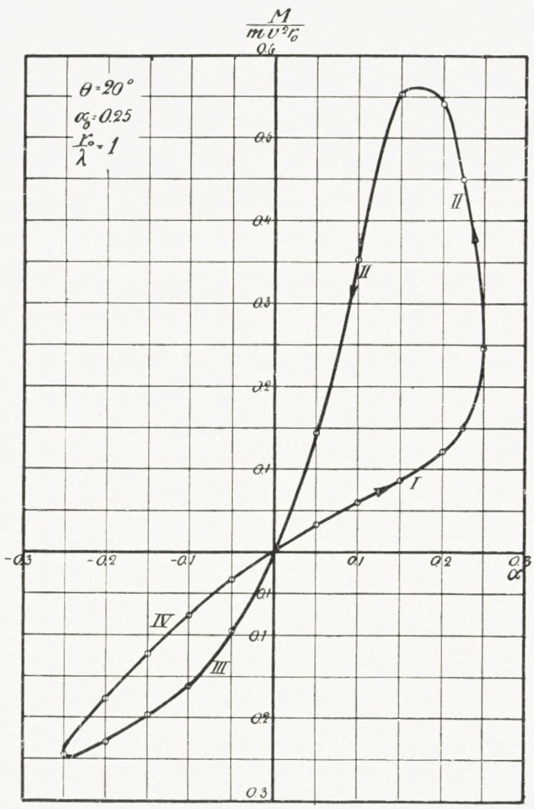


Fig. 28. $M$—$\alpha$-Curve, $\alpha_0 = 0.25$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

$\theta = 20°$, we find the three curves shown in figs. 27—29. The $\alpha - t$-curve is now single-valued, and the difference between the two time-integrals is much smaller than in the case $\alpha_0 = 0.5$, $\theta = 20°$. However, it is interesting to note that the mean values of the two integrals in the two cases only differ very slightly, the mean value in the case $\alpha_0 = 0.50$ being 28.9, while in the case $\alpha_0 = 0.25$ it is 30.5.

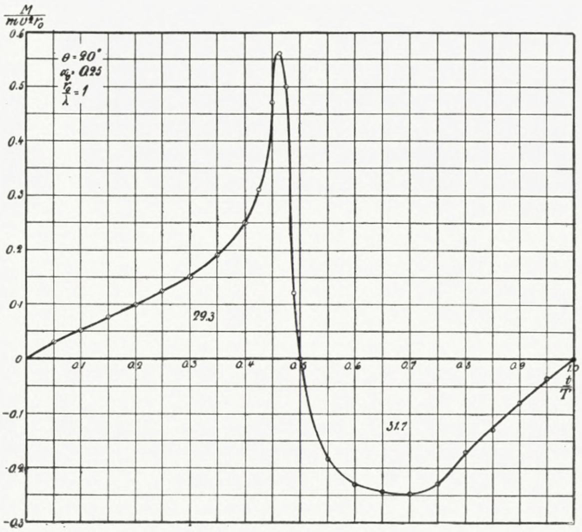


Fig. 29. $M$—$t$-Curve, $\alpha_0 = 0.25$, $\theta = 20°$, $\frac{r_0}{\lambda} = 1$.

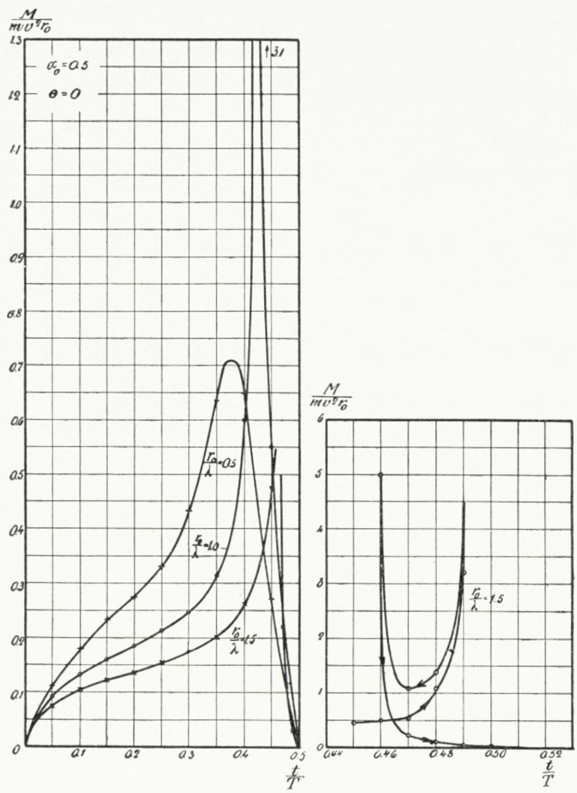


Fig. 30. $M$—$t$-Curves, $\alpha_0 = 0.5$, $\theta = 0$.

Investigations similar to those indicated above were carried out for $\theta = 0$, thus for the see-saw in its normal position. Fig. 30 shows the $M$—$t$ curves corresponding to $\alpha_0 = 0.5$, $\theta = 0$, and to $\frac{r_0}{\lambda}$ equal to 0.5, 1.0 and 1.5 respectively. One of the curves has for obvious reasons been

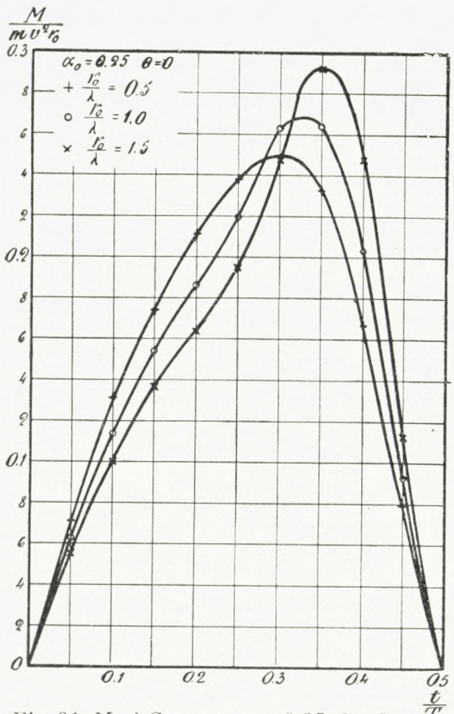


Fig. 31. $M$—$t$-Curves, $\alpha_0 = 0.25$, $\theta = 0$.

reproduced in two scales. Fig. 31 represents similar curves corresponding to the case $\alpha_0 = 0.25$, $\theta = 0$. Of course there is no longer any difference between the moments acting on the right-hand and left-hand side of the see-saw. The curves afford a direct conception of the relation between the time-integral and the parameters $\alpha_0$ and $r_0$. Thus from fig. 31 it is obvious that in the case $\alpha_0 = 0.25$ the time-integral is nearly proportional to $r_0$, the three curves representing $\frac{M}{mv^2 r_0}$ having about the same area. In the case $\alpha_0 = 0.5$ things are different. Here the time integral seems roughly independent of $r_0$, as appears from the representation in fig. 32 where the ordinate is $\frac{M}{mv^2 \lambda} = \frac{M}{mv^2 r_0} \cdot \frac{r_0}{\lambda}$. Furthermore it is seen from a comparison between fig. 30 and

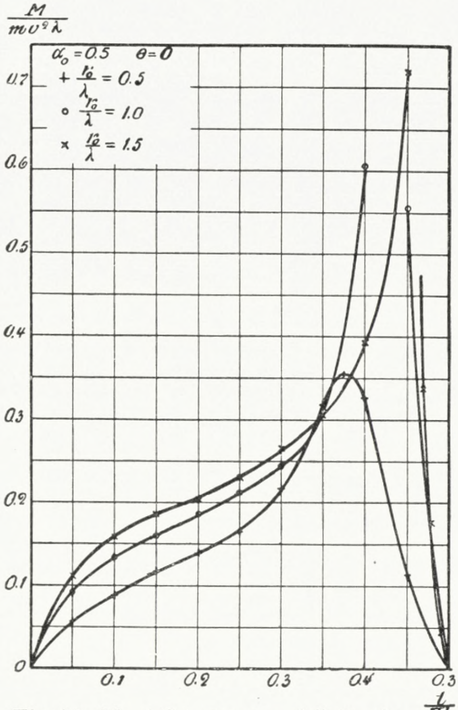


Fig. 32. $M$—$t$-Curves, $\alpha_0 = 0.5$, $\theta = 0$.

fig. 31 that for $\frac{r_0}{\lambda} = 1$ there is but a comparatively small difference between the time-integrals, the latter thus varying only slightly with $\alpha_0$.

### 3. Motion of the See-Saw under the Influence of a Jet-Wave of circular Type.

It is evident from what has been stated above that it would be practically impossible to develop an exact theory of the motion of the see-saw under the influence of the circular wave. But we may try to solve the problem approximately by replacing the actual system by some other system. We may suggest the following. Firstly we shall substitute for the wave a pure sine wave $J'$, fig. 33, travelling with a velocity $v$ in a direction under an angle $\frac{\alpha_0}{2}$ with the axis of the actual jet-wave. We shall ascribe to the jet, from which we may suppose $J'$ to originate, the mass $m'$ per cm where $m' = m \cos \frac{\alpha_0}{2}$. Now the problem of the motive moment has been reduced to that already solved in the case of a wave with constant amplitude hitting a bar deflected the angle $\frac{\alpha_0}{2}$. If $\frac{\alpha_0}{2}$ is sufficiently small the motive moment may be expressed by

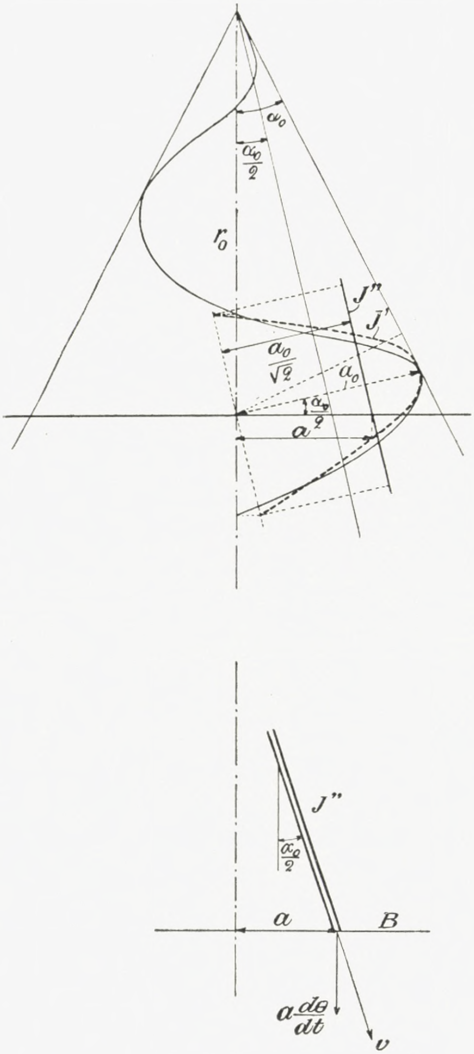

Fig. 33. Substitutes for the circular Jet-Wave.

$$(1) \qquad M = m'v^2 a_0 \sin \omega t = mv^2 a_0 \cos \frac{\alpha_0}{2} \cdot \sin \omega t\,.$$

This expression agrees with that which we may derive from

(9), paragraph 2, on the assumption of $\theta = 0$ and $\alpha_0$ being sufficiently small. For then $M = mv^2 r_0 \sin\alpha = mv^2 r_0 \sin\alpha_0 \sin\omega t$, where, compare fig. 33, $r_0 \sin\alpha_0 = a_0 \cos\frac{\alpha_0}{2}$.

Now again, when developing an expression for the damping, we shall replace the sine-wave by a jet-chain, a link of which may be $J''$, fig. 33. We shall assume the arm of the chain to be $\frac{a_0}{\sqrt{2}}$. The mass per cm of $J''$ shall be $m' = m\cos\frac{\alpha_0}{2}$. If the see-saw is moving with the angular velocity $\frac{d\theta}{dt}$ the relative velocity of $J''$ with respect to the hitting-point of the bar $B$ of the see-saw is $\left(v - a\cos\frac{\alpha_0}{2}\cdot\frac{d\theta}{dt}\right)$, fig. 33, and the force perpendicular to $B$ originating from $J''$ is

(2) $$F_n = m\left(v - a\cos\frac{\alpha_0}{2}\cdot\frac{d\theta}{dt}\right)^2 \cdot \cos^2\frac{\alpha_0}{2}.$$

Thus the moment is

(3) $$\begin{cases} M = m\left(v - a\cos\frac{\alpha_0}{2}\cdot\frac{d\theta}{dt}\right)^2 \cdot a\cos^2\frac{\alpha_0}{2} \\ \quad = mv^2 a\cos^2\frac{\alpha_0}{2} - 2\,mva^2\cos^3\frac{\alpha_0}{2}\cdot\frac{d\theta}{dt}. \end{cases}$$

Replacing $a$ by $\dfrac{a_0}{\sqrt{2}\cdot\cos\frac{\alpha_0}{2}}$ we find

(4) $$M = mv^2\frac{a_0}{\sqrt{2}}\cdot\cos\frac{\alpha_0}{2} - mva_0^2\cos\frac{\alpha_0}{2}\cdot\frac{d\theta}{dt}.$$

We thus ascribe to the actual wave the motive moment

(5) $$M = mv^2 a_0\cos\frac{\alpha_0}{2}\cdot\sin\omega t$$

and the damping moment

$$M' = -mva_0^2 \cos\frac{\alpha_0}{2} \cdot \frac{d\theta}{dt}. \tag{6}$$

If the see-saw has the moment of inertia $I$ and the directive moment $h$, then the amplitude $\theta_0$ of its stationary motion should be

$$\theta_0 = \frac{mv^2 a_0 \cos\frac{\alpha_0}{2}}{\sqrt{(h - I\omega^2)^2 + \left(mva_0^2 \cos\frac{\alpha_0}{2} \cdot \omega\right)^2}} \tag{7}$$

and if $h = I\omega^2$ (case of resonance)

$$\theta_0 = \frac{1}{2\pi} \cdot \frac{vT}{a_0} = \frac{1}{2\pi} \cdot \frac{\lambda}{a_0}. \tag{8}$$

## 4. The Turning-Moment with a Jet-Wave of rectangular Type.

Finally we shall consider a see-saw hit by a jet-wave of rectangular type, fig. 34. Such a wave or a wave of nearly that kind may be produced by oscillating the nozzle of the jet in such a manner that the axis has always the same direction. While in the case of the jet-wave of the circular type the radial velocity is always the same and equal to the velocity $v$ of the original jet, then with the rectangular type it is the velocity-component in the direction of the said jet which is equal to $v$. We proceed to develop an expression for the moment with which the jet-wave acts on the see-saw.

The element $ds$ of the wave now contains as much liquid as its projection $dx$ on the original jet, thus $m \cdot dx$. It carries with it a momentum $mv_r\,dx$, $v_r$ being the velocity in the direction of the path of the element i. e. the radius $r$. The component of the momentum perpendicular to the deflected bar $B$ is $m \cdot v_r \cdot dx \cos(\alpha + \theta)$. Now $v_r = \dfrac{v}{\cos\alpha}$ and thus the said momentum is $mv\,dx \dfrac{\cos(\alpha+\theta)}{\cos\alpha}$. This momentum is destroyed in the course of the time

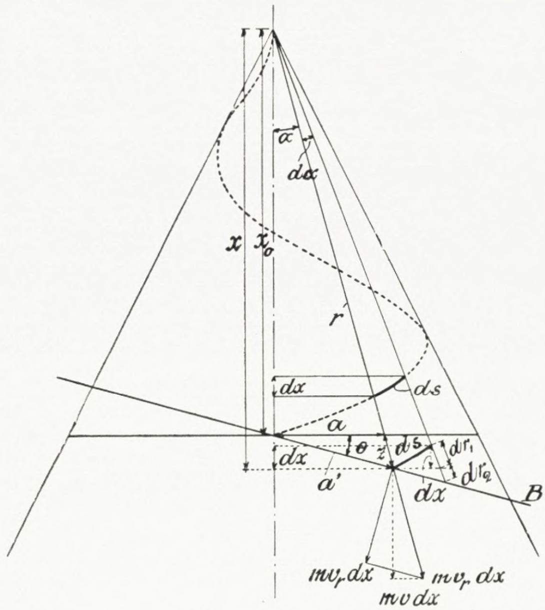


Fig. 34. See-Saw hit by a Jet-Wave of rectangular Type.

(1) $$dt = \frac{-dr_1 + dr_2}{v} \cdot \cos\alpha\,.$$

Thus the force perpendicular to $B$ and originating from the impact of the jet-wave element is

(2) $$F_n = mv^2 \frac{\cos(\alpha+\theta)}{\cos^2\alpha} \cdot \frac{dx}{-dr_1 + dr_2}\,.$$

From fig. 34 it is seen that

(3) $$dr_1 = \frac{dx}{\cos\alpha}$$

and that

(4) $$\frac{z}{\sin\theta} = \frac{a}{\cos(\alpha+\theta)} = \frac{a'}{\cos\alpha}\,.$$

From the first equation (4) we get

$$z = \sin\theta \cdot \frac{x_0\, tg\,\alpha}{\cos(\alpha+\theta)} \tag{5}$$

from which

$$dz = dr_2 = x_0 \sin\theta \left(\frac{1}{\cos^2\alpha\cos(\alpha+\theta)} + tg\,\alpha\,\frac{\sin(\alpha+\theta}{\cos^2(\alpha+\theta)}\right) d\alpha\,. \tag{6}$$

Introducing in (2) we get

$$\left\{\begin{aligned} &F_n = mv^2\,\frac{\cos(\alpha+\theta)}{\cos\alpha} \\ &\cdot\frac{1}{1 - x_0\sin\theta\left[\dfrac{1}{\cos\alpha\cos(\alpha+\theta)} + \dfrac{\sin\alpha\sin(\alpha+\theta)}{\cos^2(\alpha+\theta)}\right]\dfrac{d\alpha}{dx}}\end{aligned}\right. \tag{7}$$

and for the moment in question

$$M = F_n\, a' = F_n \cdot \frac{\cos\alpha}{\cos(\alpha+\theta)} \cdot a \tag{8}$$

thus

$$\left\{\begin{aligned} &M = mv^2\, x_0\, tg\,\alpha \\ &\cdot\frac{1}{1 - x_0\sin\theta\left[\dfrac{1}{\cos\alpha\cos(\alpha+\theta)} + \dfrac{\sin\alpha\sin(\alpha+\theta)}{\cos^2(\alpha+\theta)}\right]\dfrac{d\alpha}{dx}}.\end{aligned}\right. \tag{9}$$

The value of $\frac{d\alpha}{dx}$ must be derived from the equation of the jet-wave which at the moment $t$ is

$$tg\,\alpha = tg\,\alpha_0 \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \tag{10}$$

from which

$$\left\{\begin{aligned} \frac{d\alpha}{\cos^2\alpha} &= -\frac{2\pi}{\lambda}\, tg\,\alpha_0 \cos 2\pi\left(\frac{\mathrm{t}}{T} - \frac{x}{\lambda}\right)\cdot dx \\ &= -\frac{2\pi}{\lambda}\sqrt{tg^2\alpha_0 - tg^2\alpha}\cdot dx\end{aligned}\right. \tag{11}$$

Introducing in (9) we get

$$(12) \qquad M = mv^2 x_0 \, tg\,\alpha \cdot \frac{1}{1 + 2\pi\frac{x_0}{\lambda}\sin\theta\left[\frac{1}{\cos\alpha\cos(\alpha+\theta)} + \frac{\sin\alpha\sin(\alpha+\theta)}{\cos^2(\alpha+\theta)}\right]\sqrt{tg^2\alpha_0 - tg^2\alpha\cdot\cos^2\alpha}}.$$

From fig. 34 it is furthermore seen that

$$(13) \qquad x = x_0 + a'\sin\theta = x_0\left(1 + \sin\theta\frac{\sin\alpha}{\cos(\alpha+\theta)}\right).$$

If we introduce this in (10), we get for the relation between $\alpha$ and $t$:

$$(14) \qquad tg\,\alpha = tg\,\alpha_0 \sin 2\pi\left[\frac{t}{T} - \frac{x_0}{\lambda}\left(1 + \sin\theta\frac{\sin\alpha}{\cos(\alpha+\theta)}\right)\right].$$

If $\theta = 0$ we find from (14)

$$(15) \qquad tg\,\alpha = tg\,\alpha_0 \sin 2\pi\left(\frac{t}{T} - \frac{x_0}{\lambda}\right)$$

and from (12)

$$(16) \qquad M = mv^2 x_0\, tg\,\alpha = mv^2 x_0\, tg\,\alpha_0 \sin 2\pi\left(\frac{t}{T} - \frac{x_0}{\lambda}\right).$$

We may now carry out the same investigation as was undertaken in the case of the circular wave-type in order to learn whether in the present case we must expect the same tendency of the deflected bar to return to the position perpendicular to the axis of the jet-wave. We accordingly calculate graphs for the relations $\alpha - \frac{t}{T}$, $M - \alpha$ and $M - \frac{t}{T}$. Fig. 35 represents the $M - \frac{t}{T}$-curve for the case $\alpha_0 = 0.25$, $\theta = 20°$ and $\frac{x_0}{\lambda} = 1$. The time-integral of the moment which will carry the bar back to its normal position is but slightly greater than the time-integral which will increase the deflection. Undoubtedly the tendency in question is less pronounced than with the circular wave-type as will appear

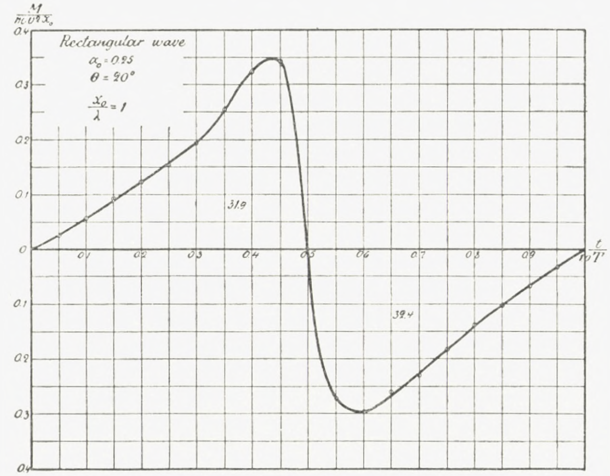


Fig. 35. $M$—$t$-Curve, $\alpha_0 = 0.25$, $\theta = 20°$, $\frac{x_0}{\lambda} = 1$.

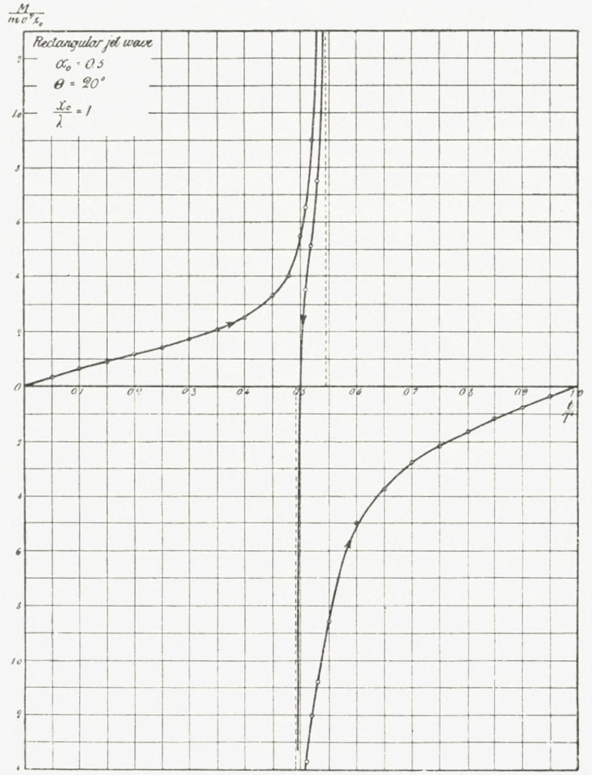


Fig. 36. $M$—$t$-Curve, $\alpha_0 = 0.50$, $\theta = 20°$, $\frac{x_0}{\lambda} = 1$.

from a comparison between fig. 29 and fig. 35. The time-integrals themselves are, however, very nearly of the same size in the two cases. In fig. 36 the $M - \frac{t}{T}$-curve corresponding to $\alpha_0 = 0.5$, $\theta = 20^\circ$ and $\frac{x_0}{\lambda} = 1$ is reproduced. Now the time-integral which tends to take the bar back to its normal position is undoubtedly appreciably greater than that — the area above the axis — which will increase the deflection.

With regard to the motion of the see-saw we shall in all probability not be far wrong if, with small values of $\alpha$, we simply replace the actual jet-wave by a wave of constant amplitude $a_0 = x_0 \, tg \, \alpha_0$ and with a velocity $v$, thus if we substitute a motive moment

(17) $$M = mv^2 a_0 \sin \omega t$$

and a damping moment

(18) $$M' = -mv a_0^2 \cdot \frac{d\theta}{dt}.$$

# CONTENTS



Færdig fra Trykkeriet den 31. Maj 1929.